

A FIRST COURSE IN LINEAR ALGEBRA

An Open Text* by Ken Kuttler

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A First Course in Linear Algebra

Ken Kuttler

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PREFACE

Linear Algebra: A First Course presents an introduction to the fascinating subject of linear algebra. As the title suggests, this text is designed as a first course in linear algebra for students who have a reasonable understanding of basic algebra. Major topics of linear algebra are presented in detail, with proofs of important theorems provided. Connections to additional topics covered in advanced courses are introduced, in an effort to assist those students who are interested in continuing on in linear algebra.

Each chapter begins with a list of desired outcomes which a student should be able to achieve upon completing the chapter. Throughout the text, examples and diagrams are given to reinforce ideas and provide guidance on how to approach various problems. Suggested exercises are given at the end of each section, and students are encouraged to work through a selection of these exercises.

A brief review of complex numbers is given, which can serve as an introduction to anyone unfamiliar with the topic.

Linear algebra is a wonderful and interesting subject, which should not be limited to a challenge of correct arithmetic. The use of a computer algebra system can be a great help in long and difficult computations. Some of the standard computations of linear algebra are easily done by the computer, including finding the reduced row-echelon form. While the use of a computer system is encouraged, it is not meant to be done without the student having an understanding of the computations.

1. SYSTEMS OF EQUATIONS

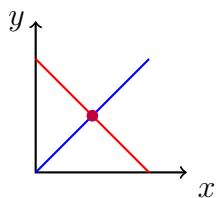
1.1 SYSTEMS OF EQUATIONS, GEOMETRY

Outcomes

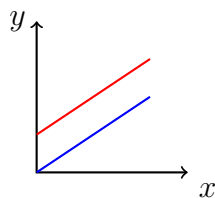
A. Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)

As you may remember, linear equations like $2x + 3y = 6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y . Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x, y) that satisfy both equations. If the point (x, y) is a point of intersection, we say that (x, y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

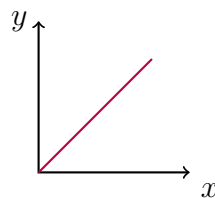
When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.



One Solution



No Solutions



Infinitely Many Solutions

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x + y = 1$ and $2x + 2y = 2$ are equations which when graphed yield the same line. In this case there are infinitely many

points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

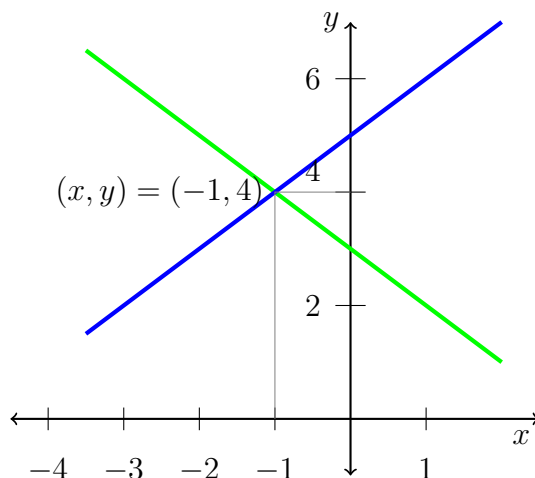
Example 1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$x + y = 3$$

$$y - x = 5$$

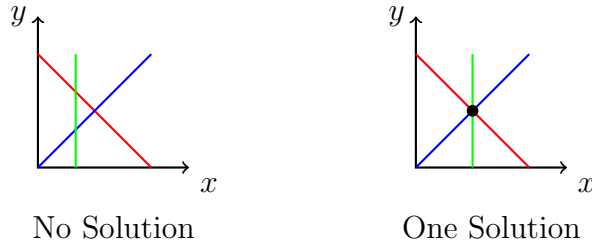
Solution. Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x, y) which satisfy both equations. In this case, there is one point of intersection at $(-1, 4)$ which means we have one unique solution, $x = -1, y = 4$.



□

In the above example, we investigated the intersection point of two equations in two variables, x and y . Now we will consider the graphical solutions of three equations in two variables.

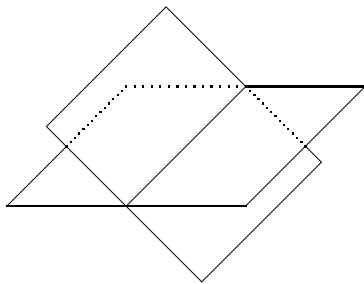
Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.



Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x, y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

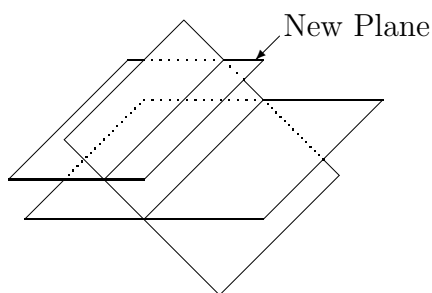
You may recall that equations in three variables, such as $2x + 4y - 5z = 8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x, y, z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.



Notice how these two planes intersect in a line. This means that the points (x, y, z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

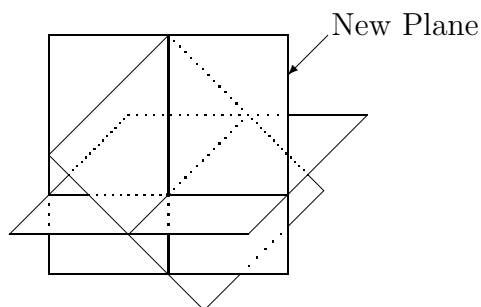
It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

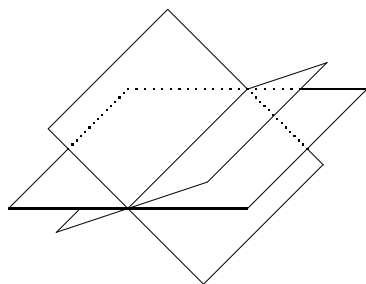


In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.



In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.



We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x + y - 2z + 4w = 8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

1.1.1. EXERCISES

1. Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.
2. Graphically, find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect.
3. You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of
 - (a) No solution.
 - (b) A unique solution.
 - (c) An infinite number of solutions.

1.2 SYSTEMS OF EQUATIONS, ALGEBRAIC PROCEDURES

Outcomes

- A. Use elementary operations to find the solution to a linear system of equations.
- B. Find the row-echelon form and reduced row-echelon form of a matrix.
- C. Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its row-echelon form.
- D. Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.
- E. Model a physical system with linear equations and then solve.

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.2: System of Linear Equations

A **system of linear equations** is a list of equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, x_1, x_2, \dots, x_n . Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, 2, 3, \dots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars**. We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i . In other words every equation equals 0. This is a special type of system.

Definition 1.3: Homogeneous System of Equations

A system of equations is called **homogeneous** if each equation in the system is equal to 0. A homogeneous system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for x_1, \dots, x_n which solve all of the equations. If such a set of values exists, we call (x_1, \dots, x_n) the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

Definition 1.4: Consistent and Inconsistent Systems

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

1.2.1. ELEMENTARY OPERATIONS

We begin this section with an example. Recall from Example 1.1 that the solution to the given system was $(x, y) = (-1, 4)$.

Example 1.5: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$\begin{aligned}x + y &= 3 \\ y - x &= 5\end{aligned}$$

Solution. By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system. \square

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

Definition 1.6: Elementary Operations

Elementary operations are those operations consisting of the following.

1. Interchange the order in which the equations are listed.
2. Multiply any equation by a nonzero number.
3. Replace any equation with itself added to a multiple of another equation.

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

Example 1.7: Effects of an Elementary Operation

Show that the system

$$\begin{aligned}x + y &= 7 \\ 2x - y &= 8\end{aligned}$$

has the same solution as the system

$$\begin{aligned}x + y &= 7 \\ -3y &= -6\end{aligned}$$

Solution. Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x - y + (-2)(x + y) = 8 + (-2)(7)$$

By simplifying, we obtain

$$-3y = -6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y = 2$. Next, we substitute this value into the first equation as follows

$$x + y = x + 2 = 7$$

Hence $x = 5$ and so $(x, y) = (5, 2)$ is a solution to the second system. We want to check if $(5, 2)$ is also a solution to the first system. We check this by substituting $(x, y) = (5, 2)$ into the system and ensuring the equations are true.

$$\begin{aligned}x + y &= (5) + (2) = 7 \\ 2x - y &= 2(5) - (2) = 8\end{aligned}$$

Hence, $(5, 2)$ is also a solution to the first system. \square

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 1.6 do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an equation, while b_i denotes a constant.

Theorem 1.8: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$\begin{aligned} E_1 &= b_1 \\ E_2 &= b_2 \end{aligned} \tag{1.1}$$

Then the following systems have the same solution set as 1.1:

1.

$$\begin{aligned} E_2 &= b_2 \\ E_1 &= b_1 \end{aligned} \tag{1.2}$$

2.

$$\begin{aligned} E_1 &= b_1 \\ kE_2 &= kb_2 \end{aligned} \tag{1.3}$$

for any scalar k , provided $k \neq 0$.

3.

$$\begin{aligned} E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1 \end{aligned} \tag{1.4}$$

for any scalar k (including $k = 0$).

Before we proceed with the proof of Theorem 1.8, let us consider this theorem in context of Example 1.7. Then,

$$\begin{aligned} E_1 &= x + y, & b_1 &= 7 \\ E_2 &= 2x - y, & b_2 &= 8 \end{aligned}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of Theorem 1.8, this action is given by

$$E_2 + (-2) E_1 = b_2 + (-2) b_1$$

or

$$2x - y + (-2)(x + y) = 8 + (-2)7$$

This gave us the second system in Example 1.7, given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2) E_1 &= b_2 + (-2) b_1 \end{aligned}$$

From this point, we were able to find the solution to the system. Theorem 1.8 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 1.8.

Proof.

1. The proof that the systems 1.1 and 1.2 have the same solution set is as follows. Suppose that (x_1, \dots, x_n) is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in 1.2 above. This is clear, because the system in 1.2 is the original system, but listed in a different order. Changing the order does not effect the solution set, so (x_1, \dots, x_n) is a solution to 1.2.
2. Next we want to prove that the systems 1.1 and 1.3 have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let (x_1, \dots, x_n) be a solution of $E_1 = b_1, E_2 = b_2$. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar k . Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if (x_1, \dots, x_n) is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, (x_1, \dots, x_n) is also a solution to 1.3.

Similarly, let (x_1, \dots, x_n) be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence (x_1, \dots, x_n) is also a solution to $E_1 = b_1, E_2 = b_2$.

3. Finally, we will prove that the systems 1.1 and 1.4 have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of 1.4. Then, we will show that any solution of 1.4 is also a solution of $E_1 = b_1, E_2 = b_2$. Let (x_1, \dots, x_n) be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in 1.4. Similarly, it also solves $E_2 = b_2$. By our proof of 1.3, it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1 , this is equal to $b_2 + kb_1$. Therefore, if (x_1, \dots, x_n) solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$.

Now suppose (x_1, \dots, x_n) solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of 1.3, it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

□

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

Example 1.9: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{aligned}x + 3y + 6z &= 25 \\2x + 7y + 14z &= 58 \\2y + 5z &= 19\end{aligned}\tag{1.5}$$

Solution. We can relate this system to Theorem 1.8 above. In this case, we have

$$\begin{aligned}E_1 &= x + 3y + 6z, & b_1 &= 25 \\E_2 &= 2x + 7y + 14z, & b_2 &= 58 \\E_3 &= 2y + 5z, & b_3 &= 19\end{aligned}$$

Theorem 1.8 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 1.6. First, replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{aligned}x + 3y + 6z &= 25 \\y + 2z &= 8 \\2y + 5z &= 19\end{aligned}\tag{1.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{aligned}x + 3y + 6z &= 25 \\y + 2z &= 8 \\z &= 3\end{aligned}\tag{1.7}$$

At this point, we can easily find the solution. Simply take $z = 3$ and substitute this back into the previous equation to solve for y , and similarly to solve for x .

$$\begin{aligned}x + 3y + 6(3) &= x + 3y + 18 = 25 \\y + 2(3) &= y + 6 = 8 \\z &= 3\end{aligned}$$

The second equation is now

$$y + 6 = 8$$

You can see from this equation that $y = 2$. Therefore, we can substitute this value into the first equation as follows:

$$x + 3(2) + 18 = 25$$

By simplifying this equation, we find that $x = 1$. Hence, the solution to this system is $(x, y, z) = (1, 2, 3)$. This process is called **back substitution**.

Alternatively, in 1.7 you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$\begin{aligned}x + 3y &= 7 \\y &= 2 \\z &= 3\end{aligned}$$

Now add (-3) times the second to the first. This yields

$$\begin{aligned}x &= 1 \\y &= 2 \\z &= 3\end{aligned}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x, y, z) = (1, 2, 3)$. \square

1.2.2. GAUSSIAN ELIMINATION

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix**. A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in Example 1.9

$$\begin{aligned}x + 3y + 6z &= 25 \\2x + 7y + 14z &= 58 \\2y + 5z &= 19\end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order,

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Similarly, we create a column from the coefficients on y in each equation, $\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$

and a column from the coefficients on z in each equation, $\begin{bmatrix} 6 \\ 14 \\ 5 \end{bmatrix}$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\begin{bmatrix} 25 \\ 58 \\ 19 \end{bmatrix}$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\begin{bmatrix} 1 & 3 & 6 & | & 25 \end{bmatrix}$ corresponds to the equation

$$x + 3y + 6z = 25.$$

Consider the following definition.

Definition 1.10: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array} \right]$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 1.6 can be used on the rows just as we used them on equations previously. Changes to a system of equations in as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem 1.8 implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 1.11: Elementary Row Operations

The **elementary row operations** (also known as **row operations**) consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by any multiple of another row added to it.

Recall how we solved Example 1.9. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Thus the first step in solving the system given by 1.5 would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Note how this corresponds to 1.6. Next take (-2) times the second row and add to the third,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as 1.7. By back substitution you obtain the solution $x = 1, y = 6$, and $z = 3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 1.12: Row-Echelon Form

An augmented matrix is in **row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any row above it.
3. Each leading entry of a row is equal to 1.

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 1.13: Reduced Row-Echelon Form

An augmented matrix is in **reduced row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any rows above it.
3. Each leading entry of a row is equal to 1.
4. All entries in a column above and below a leading entry are zero.

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

Example 1.14: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array} \right], \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Example 1.15: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

Example 1.16: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

Definition 1.17: Pivot Position and Pivot Column

A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix. A **pivot column** is a column that contains a pivot position.

For example consider the following.

Example 1.18: Pivot Position

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array} \right]$$

Where are the pivot positions and pivot columns of the augmented matrix A ?

Solution. The row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array} \right]$$

Thus the pivot columns in the matrix are the first two columns. \square

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

Algorithm 1.19: Reduced Row-Echelon Form Algorithm

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. *Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.*
2. *Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.*
3. *Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.*
4. *Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.*

The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. *Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.*

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of Algorithm 1.19.

Example 1.20: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$$

Find the row-echelon form of A . Then complete the process until A is in reduced row-echelon form.

Solution. In working through this example, we will use the steps outlined in Algorithm 1.19.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 10 & 8 \end{bmatrix}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ 10 & 8 \end{bmatrix}$$

In this matrix, the first column is a pivot column, and -5 is in the first pivot position. Therefore, we need to create a zero below it. To do this, add 2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$$

Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by -5 to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in reduced row-echelon form. \square

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

Example 1.21: Finding the Solution to a System

Give the complete solution to the following system of equations

$$\begin{aligned} 2x + 4y - 3z &= -1 \\ 5x + 10y - 7z &= -2 \\ 3x + 6y + 5z &= 9 \end{aligned}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array} \right]$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using Algorithm 1.19. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros

in the entries below the 2. First, replace the second row with -5 times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array} \right]$$

Now, replace the third row with -3 times the first row plus 2 times the third row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{array} \right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -1 times the second row and adding it to the third row yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 20 \end{array} \right]$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x + 0y + 0z = 20$$

There is no solution to this equation because for all x, y, z , the left side will equal 0 and $0 \neq 20$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent. \square

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

Example 1.22: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned} \tag{1.8}$$

Solution. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array} \right]$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Algorithm 1.19. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array} \right]$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in reduced row-echelon form, which you should verify using Definition 1.13. The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter**. Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, x, y , and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned}x &= 3 + 5(4) = 23 \\y &= 10(4) = 40 \\z &= 4\end{aligned}$$

□

In Example 1.22 the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.23: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{aligned}x + 2y - z + w &= 3 \\x + y - z + w &= 1 \\x + 3y - z + w &= 5\end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array} \right]$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Algorithm 1.19.

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array} \right]$$

Now add the second row to the third row and divide the second row by -1 .

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \tag{1.9}$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w . Assign the parameter s to z and the parameter t to w . Then the first row yields the equation $x + 2y - s + t = 3$, while the second row yields the equation $y = 2$. Since $y = 2$, the first equation becomes $x + 4 - s + t = 3$ showing that the solution is given by

$$\begin{aligned}x &= -1 + s - t \\y &= 2 \\z &= s \\w &= t\end{aligned}$$

It is customary to write this solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 + s - t \\ 2 \\ s \\ t \end{bmatrix} \quad (1.10)$$

□

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation 1.9. Instead we first place it in reduced row-echelon form as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Then the solution is $y = 2$ from the second row and $x = -1 + z - w$ from the first. Thus letting $z = s$ and $w = t$, the solution is given by 1.10.

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination**. This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the row-echelon form of the augmented matrix will have a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array} \right]$$

This row indicates that the system is inconsistent and has no solution.

- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array} \right]$$

1.2.3. UNIQUENESS OF THE REDUCED ROW-ECHELON FORM

As we have seen in earlier sections, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon form does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables x_1, x_2, \dots, x_n which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a *basic variable* whenever A has a leading 1 in column number i , in other words, when column i is a pivot column. Otherwise we say that x_i is a *free variable*.

Recall Example 1.23.

Example 1.24: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution. Recall from the solution of Example 1.23 that the row-echelon form of the augmented matrix of this system is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y , making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned}x &= -1 + s - t \\y &= 2 \\z &= s \\w &= t\end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters. \square

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 1.25: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 1.26: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon form. Then, the two systems do not have exactly the same solutions.

Proof. With respect to the linear systems associated with the matrices A and B , there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B , where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B , we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the

solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_j = -a$, while the solution of the system for matrix B has $x_j = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A , and a free variable for the other matrix B . The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j . However, by Proposition 1.25 this cannot be a solution of the system for the matrix A . This completes the proof of case 2. \square

Now, we say that the matrix B is **equivalent** to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A . The importance of this concept lies in the following result.

Theorem 1.27: Equivalent Matrices

The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

The proof of this theorem is left as an exercise.

Now, we can use Lemma 1.26 and Theorem 1.27 to prove the main result of this section.

Theorem 1.28: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof. Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A . It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+ . Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A .

Now, A^+ , B^+ , and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables x_1, x_2, \dots, x_n . Because B^+ and C^+ are each equivalent to A^+ , Theorem 1.27 ensures that all three homogeneous linear systems have exactly the same solutions. By Lemma 1.26 we conclude that $B^+ = C^+$. By construction, we must also have $B = C$. \square

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

1.2.4. RANK AND HOMOGENEOUS SYSTEMS

There is a special type of system which requires additional study. This type of system is called a homogeneous system of equations, which we defined above in Definition 1.3. Our

focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Definition 1.29: Trivial Solution

Consider the homogeneous system of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Then, $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution to this system. We call this the **trivial solution**.

If the system has a solution in which not all of the x_1, \dots, x_n are equal to zero, then we call this solution **nontrivial**. The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and $n < m$.

Consider the following example.

Example 1.30: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{aligned} 2x + y - z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Solution. Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used in the previous section. First, we construct the augmented matrix, given by

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array} \right]$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

The corresponding system of equations is

$$\begin{aligned}x &= 0 \\ y - z &= 0\end{aligned}$$

Since z is not restrained by any equation, we know that this variable will become our parameter. Let $z = t$ where t is any number. Therefore, our solution has the form

$$\begin{aligned}x &= 0 \\ y &= z = t \\ z &= t\end{aligned}$$

Hence this system has infinitely many solutions, with one parameter t . □

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{aligned}x &= 0 \\ y &= 0 + t \\ z &= 0 + t\end{aligned}$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the column of

coefficients of the parameter t . In this case, this is the column $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

There is a special name for this column, which is **basic solution**. The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions occur.

Therefore, Example 1.30 has the basic solution $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We explore this further in the following example.

Example 1.31: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$\begin{aligned}x + 4y + 3z &= 0 \\ 3x + 12y + 9z &= 0\end{aligned}$$

Find the basic solutions to this system.

Solution. The augmented matrix of this system and the resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

When written in equations, this system is given by

$$x + 4y + 3z = 0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z . Let $y = s$ and $z = t$ for any numbers s and t . Then, our solution becomes

$$\begin{aligned} x &= -4s - 3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t . Therefore, this system has two basic solutions! These are

$$X_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

□

We now present a new definition.

Definition 1.32: Linear Combination

Let X_1, \dots, X_n, V be column matrices. Then V is said to be a **linear combination** of the columns X_1, \dots, X_n if there exist scalars, a_1, \dots, a_n such that

$$V = a_1X_1 + \cdots + a_nX_n$$

A remarkable result of this section is that a linear combination of the basic solutions is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these solutions. Therefore, if we take a linear combination of the two solutions to Example 1.31, this would also be a solution. For example, we could take the following linear combination

$$3 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in Example 1.31.

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 1.33: Rank of a Matrix

*Let A be a matrix and consider any row-echelon form of A . Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** of A . We denote it by $\text{rank}(A)$.*

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A .

Example 1.34: Finding the Rank of a Matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution. First, we need to find the reduced row-echelon form of A . Through the usual algorithm, we find that this is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have two leading entries, or two pivot positions, shown above in boxes. The rank of A is $r = 2$. \square

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 1.35: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r . Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above Example 1.31 in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$ variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. Theorem 1.35 tells us that the solution will have $n - r = 3 - 1 = 2$ parameters. You can check that this is true in the solution to Example 1.31.

Notice that if $n = m$ or $n < m$, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

We are not limited to homogeneous systems of equations here. The rank of a matrix can be used to learn about the solutions of any system of linear equations. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Theorem 1.36: Rank and Solutions to a Consistent System of Equations

Let A be the $m \times (n + 1)$ augmented matrix corresponding to a consistent system of equations in n variables, and suppose A has rank r . Then

- 1. the system has a unique solution if $r = n$*
- 2. the system has infinitely many solutions if $r < n$*

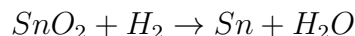
We will not present a formal proof of this, but consider the following discussions.

- 1. No Solution* The above theorem assumes that the system is consistent, that is, that it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r > 1$. Therefore, we must know that the system is consistent in order to use this theorem!
- 2. Unique Solution* Suppose $r = n$. Then, there is a pivot position in every column of the coefficient matrix of A . Hence, there is a unique solution.
- 3. Infinitely Many Solutions* Suppose $r < n$. Then there are infinitely many solutions. There are less pivot positions (and hence less leading entries) than columns, meaning that not every column is a pivot column. The columns which are *not* pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters.

1.2.5. BALANCING CHEMICAL REACTIONS

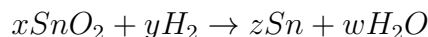
The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions.

Consider the chemical reaction



Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs and the result is a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each element we began with and how much of each element is involved in the result.

An important theory we will use here is the mass balance theory. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x, y, z, w such that



where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x, y, z, w . Thus you need

$$\begin{aligned}\text{Sn} : \quad x &= z \\ \text{O} : \quad 2x &= w \\ \text{H} : \quad 2y &= 2w\end{aligned}$$

We can rewrite these equations as

$$\begin{aligned}\text{Sn} : \quad x - z &= 0 \\ \text{O} : \quad 2x - w &= 0 \\ \text{H} : \quad 2y - 2w &= 0\end{aligned}$$

The augmented matrix for this system of equations is given by

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array} \right]$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array} \right]$$

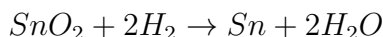
The solution is given by

$$\begin{aligned}x - \frac{1}{2}w &= 0 \\ y - w &= 0 \\ z - \frac{1}{2}w &= 0\end{aligned}$$

which we can write as

$$\begin{aligned}x &= \frac{1}{2}t \\y &= t \\z &= \frac{1}{2}t \\w &= t\end{aligned}$$

For example, let $w = 2$ and this would yield $x = 1, y = 2$, and $z = 1$. We can put these values back into the expression for the reaction which yields

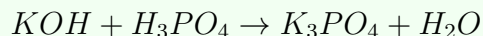


Observe that each side of the expression contains the same number of atoms of each element. This means that it preserves the total number of atoms, as required, and so the chemical reaction is balanced.

Consider another example.

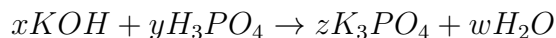
Example 1.37: Balancing a Chemical Reaction

Potassium is denoted by K , oxygen by O , phosphorus by P and hydrogen by H . Consider the reaction given by



Balance this chemical reaction.

Solution. We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that



preserves the total number of atoms of each element.

Finding these values can be done by finding the solution to the following system of equations.

$$\begin{aligned}K : \quad x &= 3z \\O : \quad x + 4y &= 4z + w \\H : \quad x + 3y &= 2w \\P : \quad y &= z\end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array} \right]$$

and the reduced row-echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

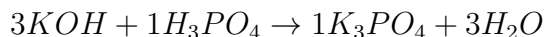
The solution is given by

$$\begin{aligned}x - w &= 0 \\y - \frac{1}{3}w &= 0 \\z - \frac{1}{3}w &= 0\end{aligned}$$

which can be written as

$$\begin{aligned}x &= t \\y &= \frac{1}{3}t \\z &= \frac{1}{3}t \\w &= t\end{aligned}$$

Choose a value for t , say 3. Then $w = 3$ and this yields $x = 3, y = 1, z = 1$. It follows that the balanced reaction is given by

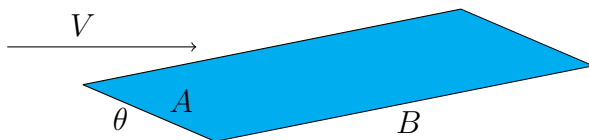


Note that this results in the same number of atoms on both sides. \square

Of course these numbers you are finding would typically be the number of moles of the molecules on each side. Thus three moles of KOH added to one mole of H_3PO_4 yields one mole of K_3PO_4 and three moles of H_2O .

1.2.6. DIMENSIONLESS VARIABLES

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown.



The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$m^0 kg^0 sec^0$
speed of wind	V	$m sec^{-1}$
speed of sound	V_0	$m sec^{-1}$
density of air	ρ	$kg m^{-3}$
viscosity	μ	$kg sec^{-1} m^{-1}$
lift	l	$kg sec^{-2} m$

Here m denotes meters, sec refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ , which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how “sticky” the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu (\text{area}) (\text{velocity gradient}) = \text{tangential force}$$

Hence

$$(\text{units on } \mu) m^2 \left(\frac{m}{\text{sec } m} \right) = kg \text{ sec}^{-2} m$$

Thus the units on μ are

$$kg \text{ sec}^{-1} m^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to feet would result in an incorrect value for l . The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f(g_1, \dots, g_k) \rho V^2 AB$$

where the units on $\rho V^2 AB$ are

$$\frac{kg}{m^3} \left(\frac{m}{\text{sec}} \right)^2 m^2 = \frac{kg \times m}{\text{sec}^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \tag{1.11}$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$m^{x_1} m^{x_2} (m^{x_4} \text{ sec}^{-x_4}) (m^{x_5} \text{ sec}^{-x_5}) (kg m^{-3})^{x_6} (kg \text{ sec}^{-1} m^{-1})^{x_7} = m^0 kg^0 \text{ sec}^0$$

Notice that there are no units on θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i .

$$\begin{aligned} m : & \quad x_1 + x_2 + x_4 + x_5 - 3x_6 - x_7 = 0 \\ \text{sec} : & \quad -x_4 - x_5 - x_7 = 0 \\ kg : & \quad x_6 + x_7 = 0 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

The reduced row-echelon form is given by

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

and so the solutions are of the form

$$x_1 = -x_2 - x_7$$

$$x_3 = x_3$$

$$x_4 = -x_5 - x_7$$

$$x_6 = -x_7$$

Thus, in terms of vectors, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -x_2 - x_7 \\ x_2 \\ x_3 \\ -x_5 - x_7 \\ x_5 \\ -x_7 \\ x_7 \end{bmatrix}$$

Thus the free variables are x_2, x_3, x_5, x_7 . By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula 1.11. For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^1$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR . Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ . Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^1$. However, it is written as V/V_0 . This is called the Mach number \mathcal{M} . Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $\text{Re} = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(\text{Re}, AR, \theta, \mathcal{M}) \text{ kg} \times \text{m} / \text{sec}^2$$

This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ . Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments to get a formula for the lift which would be reasonable. You could also consider more variables and more complicated situations in the same way.

1.2.7. EXERCISES

1. Find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$.
2. Find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$.
3. Do the three lines, $x + 2y = 1$, $2x - y = 1$, and $4x + 3y = 3$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.
4. Do the three planes, $x + y - 3z = 2$, $2x + y + z = 1$, and $3x + 2y - 2z = 0$ have a common point of intersection? If so, find one and if not, tell why there is no such point.
5. Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four people.
6. Consider the following augmented matrix in which $*$ denotes an arbitrary number and \blacksquare denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array} \right]$$

7. Consider the following augmented matrix in which $*$ denotes an arbitrary number and \blacksquare denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array} \right]$$

8. Consider the following augmented matrix in which $*$ denotes an arbitrary number and \blacksquare denotes a nonzero number. Determine whether the given augmented matrix is

consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array} \right]$$

9. Consider the following augmented matrix in which $*$ denotes an arbitrary number and \blacksquare denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array} \right]$$

10. Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.
11. If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.
12. Find h such that

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right]$$

is the augmented matrix of an *inconsistent* system.

13. Find h such that

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array} \right]$$

is the augmented matrix of a *consistent* system.

14. Find h such that

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array} \right]$$

is the augmented matrix of a *consistent* system.

15. Choose h and k such that the augmented matrix shown has each of the following:
- (a) one solution
 - (b) no solution
 - (c) infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array} \right]$$

16. Choose h and k such that the augmented matrix shown has each of the following:

- (a) one solution
- (b) no solution
- (c) infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array} \right]$$

17. Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 1 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 5 \end{aligned}$$

18. Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 0 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 3 \end{aligned}$$

19. Determine which matrices are in reduced row-echelon form.

(a) $\left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & 7 \end{array} \right]$

(b) $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$

(c) $\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right]$

20. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\left[\begin{array}{cccc} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{array} \right]$$

21. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\left[\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{array} \right]$$

22. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & -6 & -7 & -8 \\ 1 & -2 & -2 & -2 \\ 1 & -2 & -3 & -4 \end{bmatrix}$$

23. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 15 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

24. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 4 & -1 & 7 & 10 \\ 1 & 0 & 3 & 3 \\ 1 & -1 & -2 & 1 \end{bmatrix}$$

25. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

26. Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} -2 & 3 & -8 & 7 \\ 1 & -2 & 5 & -5 \\ 1 & -3 & 7 & -8 \end{bmatrix}$$

27. Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right]$$

28. Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array} \right]$$

29. Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array} \right]$$

30. Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array} \right]$$

31. Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array} \right]$$

32. Find the solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

33. Find the solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

34. Find the solution to the system of equations, $9x - 2y + 4z = -17$, $13x - 3y + 6z = -25$, and $-2x - z = 3$.

35. Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

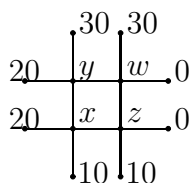
36. Find the solution to the system of equations, $8x + 2y + 3z = -3$, $8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

37. Find the solution to the system of equations, $-8x + 2y + 5z = 18$, $-8x + 3y + 5z = 13$, and $-4x + y + 5z = 19$.

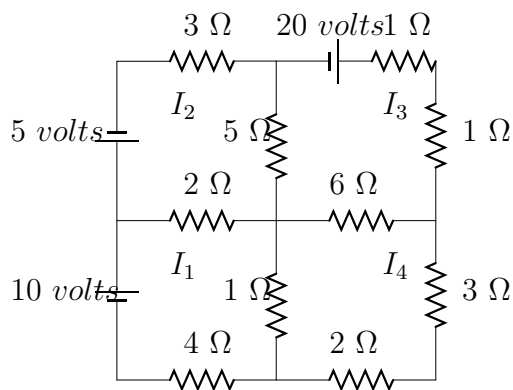
38. Find the solution to the system of equations, $3x - y - 2z = 3$, $y - 4z = 0$, and $-2x + y = -2$.

39. Find the solution to the system of equations, $-9x + 15y = 66$, $-11x + 18y = 79$, $-x + y = 4$, and $z = 3$.

40. Find the solution to the system of equations, $-19x + 8y = -108$, $-71x + 30y = -404$, $-2x + y = -12$, $4x + z = 14$.
41. Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.
42. Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.
43. Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.
44. Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?
45. The steady state temperature, u , of a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x, y, z , and w . One of the equations is $z = \frac{1}{4}(10 + 0 + w + x)$.



46. Consider the following diagram of four circuits.



The jagged lines denote resistors and the numbers next to them give their resistance in ohms, written as Ω . The breaks in the lines having one short line and one long line denote a voltage source which causes the current to flow in the direction which goes from the longer of the two lines toward the shorter along the unbroken part of the circuit. The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and

it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction. Then Kirchhoff's law states:

The sum of the resistance times the amps in the counter clockwise direction around a loop equals the sum of the voltage sources in the same direction around the loop.

In the above diagram, the top left circuit should give the equation

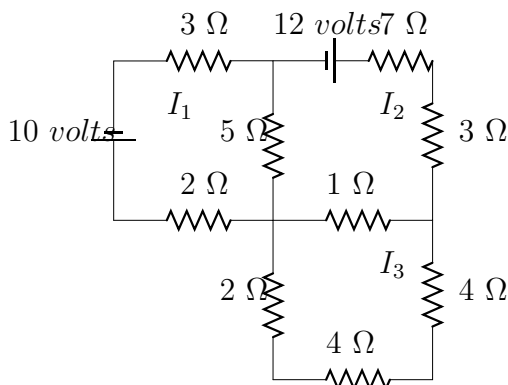
$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left, you should have

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

47. Consider the following diagram of three circuits.



The jagged lines denote resistors and the numbers next to them give their resistance in ohms, written as Ω . The breaks in the lines having one short line and one long line denote a voltage source which causes the current to flow in the direction which goes from the longer of the two lines toward the shorter along the unbroken part of the circuit. The current in amps in the four circuits is denoted by I_1, I_2, I_3 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction. Then Kirchhoff's law states:

The sum of the resistance times the amps in the counter clockwise direction around a loop equals the sum of the voltage sources in the same direction around the loop.

Find I_1, I_2, I_3 .

48. Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix}$$

49. Find the rank of the following matrix.

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

50. Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

51. Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -4 & 3 & -9 \\ 1 & -1 & 1 & -2 \\ 1 & -1 & 0 & -3 \end{bmatrix}$$

52. Find the rank of the following matrix.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix}$$

53. Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 15 & 29 \\ 1 & 4 & 8 \\ 1 & 3 & 5 \\ 3 & 9 & 15 \end{bmatrix}$$

54. Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

55. Find the rank of the following matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

56. Find the rank of the following matrix.

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix}$$

57. Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

58. Find the rank of the following matrix.

$$\begin{bmatrix} -1 & 3 & 4 & -3 & 8 \\ 1 & -3 & -4 & 2 & -5 \\ 1 & -3 & -4 & 1 & -2 \\ -2 & 6 & 8 & -2 & 4 \end{bmatrix}$$

59. Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

60. State whether each of the following sets of data are possible for the matrix equation $AX = B$. If possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, $[A|B]$ denotes the augmented matrix.

(a) A is a 5×6 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.

(b) A is a 3×4 matrix, $\text{rank}(A) = 3$ and $\text{rank}[A|B] = 2$.

(c) A is a 4×2 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.

(d) A is a 5×5 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 5$.

(e) A is a 4×2 matrix, $\text{rank}(A) = 2$ and $\text{rank}[A|B] = 2$.

61. Consider the system $-5x + 2y - z = 0$ and $-5x - 2y - z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Does it follow that x and z can equal anything? Notice that when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, the equations do not equal 0. Why?
62. Balance the following chemical reactions.
- (a) $KNO_3 + H_2CO_3 \rightarrow K_2CO_3 + HNO_3$
 - (b) $AgI + Na_2S \rightarrow Ag_2S + NaI$
 - (c) $Ba_3N_2 + H_2O \rightarrow Ba(OH)_2 + NH_3$
 - (d) $CaCl_2 + Na_3PO_4 \rightarrow Ca_3(PO_4)_2 + NaCl$
63. In the section on dimensionless variables it was observed that $\rho V^2 AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables which will yield something which has the units of force.

2. MATRICES

2.1 MATRIX ARITHMETIC

Outcomes

- A. Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
- B. Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.
- C. Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
- E. Solve a linear system using matrix algebra.
- F. Use multiplication by an elementary matrix to apply row operations.
- G. Write a matrix as a product of elementary matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \quad (2.1)$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns. You might remember that you always list the rows before the columns by using the phrase **Row**man **Cath**olic.

Consider the following definition.

Definition 2.1: Square Matrix

A matrix A which has size $n \times n$ is called a **square matrix**. In other words, A is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

Therefore, A_j is the j^{th} column of A , when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A . Elements of the matrix are identified according to their position. The **(i,j)-entry** of a matrix is the entry in the i^{th} row and j^{th} column. For example, in the matrix 2.1 above, 8 is in position (2,3) (and is called the (2,3)-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij} . Then, we can write A in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix in 2.1, $a_{23} = 8$, $a_{32} = -9$, $a_{12} = 2$, etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 2.2: The Zero Matrix

The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by 0 .

One possible zero matrix is shown in the following example.

Example 2.3: The Zero Matrix

The 2×3 zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

Definition 2.4: Equality of Matrices

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

2.1.1. ADDITION OF MATRICES

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, as one has size 3×2 while the other has size 2×3 .

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix}$$

is possible.

The formal definition is as follows.

Definition 2.5: Addition of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

Example 2.6: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution. Notice that both A and B are of size 2×3 . Since A and B are of the same size, the addition is possible. Using Definition 2.5, the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}$$

□

Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A + B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Proposition 2.7: Properties of Matrix Addition

Let A, B and C be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A + B = B + A \quad (2.2)$$

- Associative Law of Addition

$$(A + B) + C = A + (B + C) \quad (2.3)$$

- Existence of an Additive Identity

$$\begin{aligned} &\text{There exists a zero matrix } 0 \text{ such that} \\ &A + 0 = A \end{aligned} \quad (2.4)$$

- Existence of an Additive Inverse

$$\begin{aligned} &\text{There exists a matrix } -A \text{ such that} \\ &A + (-A) = 0 \end{aligned} \quad (2.5)$$

Proof. Consider the Commutative Law of Addition given in 2.2. Let A, B, C , and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in Definition 2.5. Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the ij^{th} entries are the same for all i and j . Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise. \square

We call the zero matrix in 2.4 the **additive identity**. Similarly, we call the matrix $-A$ in 2.5 the **additive inverse**. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1 . In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)A$.

2.1.2. SCALAR MULTIPLICATION OF MATRICES

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

Definition 2.8: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Consider the following example.

Example 2.9: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Solution. By Definition 2.8, we multiply each element of A by 7. Therefore,

$$7A = 7 \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

\square

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

Proposition 2.10: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- *Distributive Law over Matrix Addition*

$$k(A + B) = kA + kB$$

- *Distributive Law over Scalar Addition*

$$(k + p)A = kA + pA$$

- *Associative Law for Scalar Multiplication*

$$k(pA) = (kp)A$$

- *Rule for Multiplication by 1*

$$1A = A$$

The proof of this proposition is similar to the proof of Proposition 2.7 and is left an exercise to the reader.

2.1.3. MULTIPLICATION OF MATRICES

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

Definition 2.11: Row and Column Vectors

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If X is such a matrix, then we write x_i to denote the entry of X in the i^{th} row of a column matrix, or the i^{th} column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector**. The $1 \times n$ matrix

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

In this chapter, we will again use the notion of linear combination of vectors as in Definition 1.32. In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 2.12: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

*We can express this system in **vector form** which is as follows:*

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

In general terms,

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} \end{aligned}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 2.13: Multiplication of Vector by Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A :

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \sum_{j=1}^n x_j A_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

Example 2.14: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution. We will use Definition 2.13 to compute the product. Therefore, we compute the product AX as follows.

$$\begin{aligned} & 1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned}$$

□

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 2.15: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This is also known as **The Form** $AX = B$. The matrix A is simply the coefficient matrix of the system. The vector X is the column vector constructed from the variables of

the system, and the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0 .

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries x_1, x_2, \dots, x_n of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in Example 2.14, we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B , in order to form the product AB , the number of columns of A must equal the number of rows of B . Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n}^{\text{these must match!}} (n \times p)) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 2.16: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where B_1, \dots, B_p are the $n \times 1$ columns of B . Then the $m \times p$ matrix AB is defined as follows:

$$AB = A[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the k^{th} column of AB .

Consider the following example.

Example 2.17: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution. The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3 . The inside numbers are equal, so A and B are conformable matrices. According to the above discussion AB will be a 2×3 matrix. Definition 2.16 gives us a way to calculate each column of AB , as follows.

$$\left[\overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{Third column}} \right]$$

You know how to multiply a matrix times a vector, using Definition 2.13 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

□

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

Example 2.18: Vector Times Vector Multiplication

Multiply if possible $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$.

Solution. In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4 . The inside numbers match so the product is defined. Note that the product will be a matrix of size 3×4 . Using Definition 2.16, we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix} = \left[\overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{Third column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}^{\text{Fourth column}} \right]$$

You can use Definition 2.13 to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

□

Example 2.19: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. First check if it is possible. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so you can't do this multiplication. □

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 2.17. In this example, we tried to calculate BA instead of AB . This demonstrates another property of matrix multiplication. While the product AB maybe be defined, we cannot assume that the product BA will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Example 2.20: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$. □

Notice that we could also multiply A by the 2×1 zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

2.1.4. THE ij^{th} ENTRY OF A PRODUCT

In previous sections, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij^{th} entry of AB ? It is the entry in the i^{th} row and the j^{th} column of the product AB .

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

The j^{th} column of AB is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the ij^{th} entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

The following is the formal definition for the ij^{th} entry of a product of matrices.

Definition 2.21: The ij^{th} Entry of a Product

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j) -entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j) -entry of the product AB , or $(AB)_{ij}$, you multiply the i^{th} row of A , on the left by the j^{th} column of B . To express AB in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

Example 2.22: The Entries of a Product

Compute AB if possible. If it is, find the $(3, 2)$ -entry of AB using Definition 2.21.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

Solution. First check if the product is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB :

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{bmatrix}$$

which is a 3×3 matrix as desired. Thus, the $(3, 2)$ -entry equals 42.

Now using Definition 2.21, we can find that the $(3, 2)$ -entry equals

$$\begin{aligned}\sum_{k=1}^2 a_{3k}b_{k2} &= a_{31}b_{12} + a_{32}b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42\end{aligned}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct.

□

Here is another example.

Example 2.23: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2, 1)$ -entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$$

Solution. This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2, 1)$ -entry of AB , that is, the entry in the second row and first column of the product. We will use Definition 2.21, which states

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2, 1)$ -entry is found by computing

$$(AB)_{21} = \sum_{k=1}^3 a_{2k}b_{k1} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB . You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

□

2.1.5. PROPERTIES OF MATRIX MULTIPLICATION

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both AB and BA are defined, they may not be equal.

Example 2.24: Matrix Multiplication is Not Commutative

Compare the products AB and BA , for matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution. First, notice that A and B are both of size 2×2 . Therefore, both products AB and BA are defined. The first product, AB is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The second product, BA is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Therefore, $AB \neq BA$. □

This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices A and B it is true that $AB = BA$, then we say that A and B **commute**. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 2.25: Properties of Matrix Multiplication

The following hold for matrices A, B , and C and for scalars r and s ,

- $$A(rB + sC) = r(AB) + s(AC) \quad (2.6)$$

- $$(B + C)A = BA + CA \quad (2.7)$$

- $$A(BC) = (AB)C \quad (2.8)$$

Proof. First we will prove 2.6. We will use Definition 2.21 and prove this statement using the ij^{th} entries of a matrix. Therefore,

$$(A(rB + sC))_{ij} = \sum_k a_{ik} (rB + sC)_{kj} = \sum_k a_{ik} (rb_{kj} + sc_{kj})$$

$$\begin{aligned}
&= r \sum_k a_{ik} b_{kj} + s \sum_k a_{ik} c_{kj} = r (AB)_{ij} + s (AC)_{ij} \\
&= (r (AB) + s (AC))_{ij}
\end{aligned}$$

Thus $A(rB + sC) = r(AB) + s(AC)$ as claimed.

The proof of 2.7 follows the same pattern and is left as an exercise.

Statement 2.8 is the associative law of multiplication. Using Definition 2.21,

$$\begin{aligned}
(A(BC))_{ij} &= \sum_k a_{ik} (BC)_{kj} = \sum_k a_{ik} \sum_l b_{kl} c_{lj} \\
&= \sum_l (AB)_{il} c_{lj} = ((AB)C)_{ij}.
\end{aligned}$$

This proves 2.8. \square

2.1.6. THE TRANSPOSE

Another important operation on matrices is that of taking the **transpose**. For a matrix A , we denote the **transpose** of A by A^T . Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 2.26: The Transpose of a Matrix

Let A be an $m \times n$ matrix. Then A^T , the **transpose** of A , denotes the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i, j) -entry of A becomes the (j, i) -entry of A^T . Consider the following example.

Example 2.27: The Transpose of a Matrix

Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Solution. By Definition 2.26, we know that for $A = [a_{ij}]$, $A^T = [a_{ji}]$. In other words, we switch the row and column location of each entry. The $(1, 2)$ -entry becomes the $(2, 1)$ -entry.

Thus,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Notice that A is a 2×3 matrix, while A^T is a 3×2 matrix. □

The transpose of a matrix has the following important properties .

Lemma 2.28: Properties of the Transpose of a Matrix

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars. Then

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(rA + sB)^T = rA^T + sB^T$

Proof. First we prove 2. From Definition 2.26,

$$\begin{aligned} (AB)^T &= [(AB)_{ij}]^T = [(AB)_{ji}] = \sum_k a_{jk} b_{ki} = \sum_k b_{ki} a_{jk} \\ &= \sum_k [b_{ki}]^T [a_{jk}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T \end{aligned}$$

The proof of Formula 3 is left as an exercise. □

The transpose of a matrix is related to other important topics. Consider the following definition.

Definition 2.29: Symmetric and Skew Symmetric Matrices

An $n \times n$ matrix A is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

We will explore these definitions in the following examples.

Example 2.30: Symmetric Matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 2.29 to show that A is symmetric.

Solution. By Definition 2.29, we need to show that $A = A^T$. Now, using Definition 2.26,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric. \square

Example 2.31: A Skew Symmetric Matrix

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Solution. By Definition 2.29,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of A^T is equal to -1 times the same entry of A . Hence, $A^T = -A$ and so by Definition 2.29, A is skew symmetric. \square

2.1.7. THE IDENTITY AND INVERSES

There is a special matrix, denoted I , which is called to as the **identity matrix**. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere. Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the ij^{th} entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where δ_{ij} is the **Kronecker symbol** defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

I_n is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

Lemma 2.32: Multiplication by the Identity Matrix

Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Then $AI_n = A$. If I_m is the $m \times m$ identity matrix, it also follows that $I_mA = A$.

Proof. The (i, j) -entry of AI_n is given by:

$$\sum_k a_{ik} \delta_{kj} = a_{ij}$$

and so $AI_n = A$. The other case is left as an exercise for you. \square

We now define the matrix operation which in some ways plays the role of division.

Definition 2.33: The Inverse of a Matrix

A square $n \times n$ matrix A is said to have an **inverse** A^{-1} if and only if

$$AA^{-1} = A^{-1}A = I_n$$

In this case, the matrix A is called **invertible**.

Such a matrix A^{-1} will have the same size as the matrix A . It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it **is** the inverse.

Theorem 2.34: Uniqueness of Inverse

Suppose A is an $n \times n$ matrix such that an inverse A^{-1} exists. Then there is only one such inverse matrix. That is, given any matrix B such that $AB = BA = I$, $B = A^{-1}$.

Proof. In this proof, it is assumed that I is the $n \times n$ identity matrix. Let A, B be $n \times n$ matrices such that A^{-1} exists and $AB = BA = I$. We want to show that $A^{-1} = B$. Now using properties we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique. \square

The next example demonstrates how to check the inverse of a matrix.

Example 2.35: Verifying the Inverse of a Matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A .

Solution. To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

showing that this matrix is indeed the inverse of A . \square

Unlike ordinary multiplication of numbers, it can happen that $A \neq 0$ but A may fail to have an inverse. This is illustrated in the following example.

Example 2.36: A Nonzero Matrix With No Inverse

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that A does not have an inverse.

Solution. One might think A would have an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If A^{-1} existed, we would have the following

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^{-1} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= A^{-1} \left(A \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \\ &= (A^{-1}A) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= I \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

This says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is impossible! Therefore, A does not have an inverse. \square

In the next section, we will explore how to find the inverse of a matrix, if it exists.

2.1.8. FINDING THE INVERSE OF A MATRIX

In Example 2.35, we were given A^{-1} and asked to verify that this matrix was in fact the inverse of A . In this section, we explore how to find A^{-1} .

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

as in Example 2.35. In order to find A^{-1} , we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array} \right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array} \right] \tag{2.9}$$

for the second.

Let's solve the first system. Take -1 times the first row and add to the second to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array} \right]$$

Now take -1 times the second row and add to the first to get

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right]$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, 2.9 to find z and w . You will find that $z = -1$ and $w = 1$.

If we take the values found for x, y, z , and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

After taking the time to solve the second system, you may have noticed that exactly the same row operations were used to solve both systems. In each case, the end result was something of the form $[I|X]$ where I is the identity and X gave a column of the inverse. In the above,

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\begin{bmatrix} z \\ w \end{bmatrix}$$

To simplify this procedure, we could have solved both systems at once! To do so, we could have written

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right]$$

and row reduced until we obtained

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

and read off the inverse as the 2×2 matrix on the right side.

This exploration motivates the following important algorithm.

Algorithm 2.37: Matrix Inverse Algorithm

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. In this case, we say that A is **invertible**. If it is impossible to row reduce to a matrix of the form $[I|B]$, then A has no inverse.

This algorithm shows how to find the inverse if it exists. It will also tell you if A does not have an inverse.

Consider the following example.

Example 2.38: Finding the Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find A^{-1} if it exists.

Solution. Set up the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right]$$

Now we row reduce, with the goal of obtaining the 3×3 identity matrix on the left hand side. First, take -1 times the first row and add to the second followed by -3 times the first row added to the third row. This yields

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array} \right]$$

Then take 5 times the second row and add to -2 times the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Next take the third row and add to -7 times the first row. This yields

$$\left[\begin{array}{ccc|ccc} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\left[\begin{array}{ccc|ccc} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array} \right]$$

Finally divide the first row by -7, the second row by -10 and the third row by 14 which yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array} \right]$$

Notice that the left hand side of this matrix is now the 3×3 identity matrix I_3 . Therefore, the inverse is the 3×3 matrix on the right hand side, given by

$$\left[\begin{array}{ccc} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array} \right]$$

□

It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

Example 2.39: A Matrix Which Has No Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find A^{-1} if it exists.

Solution. Write the augmented matrix $[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array} \right]$$

and proceed to do row operations attempting to obtain $[I|A^{-1}]$. Take -1 times the first row and add to the second. Then take -2 times the first row and add to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array} \right]$$

Next add -1 times the second row to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array} \right]$$

At this point, you can see there will be no way to obtain I on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of A does not exist. In this case, we say that A is not invertible. □

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.35. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated A^{-1} properly!

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.15 that we can write a system of equations in matrix form, which is of the form $AX = B$. Suppose you find the inverse of the matrix A^{-1} . Then you could multiply both sides of this equation on the left by A^{-1} and simplify to obtain

$$\begin{aligned} (A^{-1})AX &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find X , the solution to the system, by computing $X = A^{-1}B$. Note that once you have found A^{-1} , you can easily get the solution for different right hand sides (different B). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

Example 2.40: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned}x + z &= 1 \\x - y + z &= 3 \\x + y - z &= 2\end{aligned}$$

Solution. First, we can write the system of equations in matrix form

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = B \quad (2.10)$$

The inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system 2.10 is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$

□

What if the right side, B , of 2.10 had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where A^{-1} exists, it is easy to find the solution when the vector B is changed.

We conclude this section with some important properties of the inverse.

Theorem 2.41: Inverses of Transposes and Products

Let A, B , and A_i for $i = 1, \dots, k$ be $n \times n$ matrices.

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$
2. If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A_1, A_2, \dots, A_k are invertible, then the product $A_1A_2 \cdots A_k$ is invertible, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$

Consider the following theorem.

Theorem 2.42: Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix.

1. I is invertible and $I^{-1} = I$
2. If A is invertible then so is A^{-1} , and $(A^{-1})^{-1} = A$
3. If A is invertible then so is A^k , and $(A^k)^{-1} = (A^{-1})^k$
4. If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

2.1.9. ELEMENTARY MATRICES

We now turn our attention to a special type of matrix called an **elementary matrix**. An elementary matrix is always a square matrix. Recall the row operations given in Definition 1.11. Any elementary matrix, which we often denote by E , is obtained from applying *one* row operation to the identity matrix of the same size.

For example, the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the elementary matrix obtained from switching the two rows. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the elementary matrix obtained from multiplying the second row of the 3×3 identity matrix by 3. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

is the elementary matrix obtained from adding -3 times the first row to the third row.

You may construct an elementary matrix from any row operation, but remember that you can only apply one operation.

Consider the following definition.

Definition 2.43: Elementary Matrices and Row Operations

Let E be an $n \times n$ matrix. Then E is an **elementary matrix** if it is the result of applying one row operation to the $n \times n$ identity matrix I_n .

Those which involve switching rows of the identity matrix are called *permutation matrices*.

Therefore, E constructed above by switching the two rows of I_2 is called a permutation matrix.

Elementary matrices can be used in place of row operations and therefore are very useful. It turns out that multiplying (on the left hand side) by an elementary matrix E will have the same effect as doing the row operation used to obtain E .

The following theorem is an important result which we will use throughout this text.

Theorem 2.44: Multiplication by an Elementary Matrix and Row Operations

To perform any of the three row operations on a matrix A it suffices to take the product EA , where E is the elementary matrix obtained by using the desired row operation on the identity matrix.

Therefore, instead of performing row operations on a matrix A , we can row reduce through matrix multiplication with the appropriate elementary matrix. We will examine this theorem in detail for each of the three row operations given in Definition 1.11.

First, consider the following lemma.

Lemma 2.45: Action of Permutation Matrix

Let P^{ij} denote the elementary matrix which involves switching the i^{th} and the j^{th} rows. Then P^{ij} is a permutation matrix and

$$P^{ij} A = B$$

where B is obtained from A by switching the i^{th} and the j^{th} rows.

We will explore this idea more in the following example.

Example 2.46: Switching Rows with an Elementary Matrix

Let

$$P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ g & d \\ e & f \end{bmatrix}$$

Find B where $B = P^{12} A$.

Solution. You can see that the matrix P^{12} is obtained by switching the first and second rows of the 3×3 identity matrix I .

Using our usual procedure, compute the product $P^{12} A = B$. The result is given by

$$B = \begin{bmatrix} g & d \\ a & b \\ e & f \end{bmatrix}$$

Notice that B is the matrix obtained by switching rows 1 and 2 of A . Therefore by multiplying A by P^{12} , the row operation which was applied to I to obtain P^{12} is applied to A to obtain B . \square

Theorem 2.44 applies to all three row operations, and we now look at the row operation of multiplying a row by a scalar. Consider the following lemma.

Lemma 2.47: Multiplication by a Scalar and Elementary Matrices

Let $E(k, i)$ denote the elementary matrix corresponding to the row operation in which the i^{th} row is multiplied by the nonzero scalar, k . Then

$$E(k, i) A = B$$

where B is obtained from A by multiplying the i^{th} row of A by k .

We will explore this lemma further in the following example.

Example 2.48: Multiplication of a Row by 5 Using Elementary Matrix

Let

$$E(5, 2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find the matrix B where $B = E(5, 2)A$

Solution. You can see that $E(5, 2)$ is obtained by multiplying the second row of the identity matrix by 5.

Using our usual procedure for multiplication of matrices, we can compute the product $E(5, 2)A$. The resulting matrix is given by

$$B = \begin{bmatrix} a & b \\ 5c & 5d \\ e & f \end{bmatrix}$$

Notice that B is obtained by multiplying the second row of A by the scalar 5. □

There is one last row operation to consider. The following lemma discusses the final operation of adding a multiple of a row to another row.

Lemma 2.49: Adding Multiples of Rows and Elementary Matrices

Let $E(k \times i + j)$ denote the elementary matrix obtained from I by adding k times the i^{th} row to the j^{th} . Then

$$E(k \times i + j)A = B$$

where B is obtained from A by adding k times the i^{th} row to the j^{th} row of A .

Consider the following example.

Example 2.50: Adding Two Times the First Row to the Last

Let

$$E(2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find B where $B = E(2 \times 1 + 3)A$.

Solution. You can see that the matrix $E(2 \times 1 + 3)$ was obtained by adding 2 times the first row of I to the third row of I .

Using our usual procedure, we can compute the product $E(2 \times 1 + 3)A$. The resulting matrix B is given by

$$B = \begin{bmatrix} a & b \\ c & d \\ 2a + e & 2b + f \end{bmatrix}$$

You can see that B is the matrix obtained by adding 2 times the first row of A to the third row. \square

Suppose we have applied a row operation to a matrix A . Consider the row operation required to return A to its original form, to undo the row operation. It turns out that this action is how we find the inverse of an elementary matrix E .

Consider the following theorem.

Theorem 2.51: Elementary Matrices and Inverses

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I . E^{-1} will be obtained by performing the row operation which would carry E back to I .

- If E is obtained by switching rows i and j , then E^{-1} is also obtained by switching rows i and j .
- If E is obtained by multiplying row i by the scalar k , then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j , then E^{-1} is obtained by subtracting k times row i from row j .

Consider the following example.

Example 2.52: Inverse of an Elementary Matrix

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Find E^{-1} .

Solution. Consider the elementary matrix E given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence, E^{-1} is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

We can verify that $EE^{-1} = I$. Take the product EE^{-1} , given by

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This equals I so we know that we have compute E^{-1} properly. \square

Suppose an $m \times n$ matrix A is row reduced to its reduced row-echelon form. By tracking each row operation completed, this row reduction can be completed through multiplication by elementary matrices. Consider the following definition.

Definition 2.53: The Form $B = UA$

Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A . Then we can write $B = UA$ where U is the product of all elementary matrices representing the row operations done to A to obtain B .

Consider the following example.

Example 2.54: The Form $B = UA$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Find B , the reduced row-echelon form of A and write it in the form $B = UA$.

Solution. To find B , row reduce A . For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The resulting matrix is equivalent to finding the product of $P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and A .

Next, add (-2) times row 1 to row 3.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is equivalent to multiplying by the matrix $E(-2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Notice that the resulting matrix is B , the required reduced row-echelon form of A .

We can then write

$$\begin{aligned} B &= E(-2 \times 1 + 2) (P^{12}A) \\ &= (E(-2 \times 1 + 2)P^{12}) A \\ &= UA \end{aligned}$$

It remains to find the matrix U .

$$\begin{aligned}
 U &= E(-2 \times 1 + 2)P^{12} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}
 \end{aligned}$$

We can verify that $B = UA$ holds for this matrix U :

$$\begin{aligned}
 UA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
 &= B
 \end{aligned}$$

□

While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to B . The following theorem provides an alternate way to find the matrix U .

Theorem 2.55: Finding the Matrix U

Let A be an $m \times n$ matrix and let B be its reduced row-echelon form. Then $B = UA$ where U is an invertible $m \times m$ matrix found by forming the matrix $[A|I_m]$ and row reducing to $[B|U]$.

Let's revisit the above example using the process outlined in Theorem 2.55.

Example 2.56: The Form $B = UA$, Revisited

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Using the process outlined in Theorem 2.55, find U such that $B = UA$.

Solution. First, set up the matrix $[A|I_m]$.

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A .

$$\begin{aligned} \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array} \right] \end{aligned}$$

The left side of this matrix is B , and the right side is U . Comparing this to the matrix U found above in Example 2.54, you can see that the same matrix is obtained regardless of which process is used. \square

Recall from Algorithm 2.37 that an $n \times n$ matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using the usual row operations. This leads to an important consequence related to the above discussion.

Suppose A is an $n \times n$ invertible matrix. Then, set up the matrix $[A|I_n]$ as done above, and row reduce until it is of the form $[B|U]$. In this case, $B = I_n$ because A is invertible.

$$\begin{aligned} B &= UA \\ I_n &= UA \\ U^{-1} &= A \end{aligned}$$

Now suppose that $U = E_1 E_2 \cdots E_k$ where each E_i is an elementary matrix representing a row operation used to carry A to I . Then,

$$U^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

Remember that if E_i is an elementary matrix, so too is E_i^{-1} . It follows that

$$\begin{aligned} A &= U^{-1} \\ &= E_k^{-1} \cdots E_2^{-1} E_1^{-1} \end{aligned}$$

and A can be written as a product of elementary matrices.

Theorem 2.57: Product of Elementary Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Consider the following example.

Example 2.58: Product of Elementary Matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Write A as a product of elementary matrices.

Solution. We will use the process outlined in Theorem 2.55 to write A as a product of elementary matrices. We will set up the matrix $[A|I]$ and row reduce, recording each row operation as an elementary matrix.

First:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Secondly:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Notice that the reduced row-echelon form of A is I . Hence $I = UA$ where U is the product of the above elementary matrices. It follows that $A = U^{-1}$. Since we want to write A as a product of elementary matrices, we wish to express U^{-1} as a product of elementary matrices.

$$\begin{aligned} U^{-1} &= (E_3 E_2 E_1)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= A \end{aligned}$$

This gives A written as a product of elementary matrices. By Theorem 2.57 it follows that A is invertible. \square

2.1.10. MORE ON MATRIX INVERSES

In this section, we will prove three theorems which will clarify the concept of matrix inverses. In order to do this, first recall some important properties of elementary matrices.

Recall that an elementary matrix is a square matrix obtained by performing an elementary operation on an identity matrix. Each elementary matrix is invertible, and its inverse is also an elementary matrix. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the product EA is the result of applying to A the same elementary row operation that was applied to the $m \times m$ identity matrix in order to obtain E .

Let R be the reduced row-echelon form of an $m \times n$ matrix A . R is obtained by iteratively applying a sequence of elementary row operations to A . Denote by E_1, E_2, \dots, E_k the elementary matrices associated with the elementary row operations which were applied, in order, to the matrix A to obtain the resulting R . We then have that $R = (E_k \cdots (E_2 (E_1 A))) = E_k \cdots E_2 E_1 A$. Let E denote the product matrix $E_k \cdots E_2 E_1$ so that we can write $R = EA$ where E is an invertible matrix whose inverse is the product $(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$.

Now, we will consider some preliminary lemmas.

Lemma 2.59: Invertible Matrix and Zeros

Suppose that A and B are matrices such that the product AB is an identity matrix. Then the reduced row-echelon form of A does not have a row of zeros.

Proof: Let R be the reduced row-echelon form of A . Then $R = EA$ for some invertible square matrix E as described above. By hypothesis $AB = I$ where I is an identity matrix, so we have a chain of equalities

$$R(BE^{-1}) = (EA)(BE^{-1}) = E(AB)E^{-1} = EIE^{-1} = EE^{-1} = I$$

If R would have a row of zeros, then so would the product $R(BE^{-1})$. But since the identity matrix I does not have a row of zeros, neither can R have one. ■

We now consider a second important lemma.

Lemma 2.60: Size of Invertible Matrix

Suppose that A and B are matrices such that the product AB is an identity matrix. Then A has at least as many columns as it has rows.

Proof: Let R be the reduced row-echelon form of A . By Lemma 2.59, we know that R does not have a row of zeros, and therefore each row of R has a leading 1. Since each column of R contains at most one of these leading 1s, R must have at least as many columns as it has rows. ■

An important theorem follows from this lemma.

Theorem 2.61: Invertible Matrices are Square

Only square matrices can be invertible.

Proof: Suppose that A and B are matrices such that both products AB and BA are identity matrices. We will show that A and B must be square matrices of the same size. Let the matrix A have m rows and n columns, so that A is an $m \times n$ matrix. Since the product

AB exists, B must have n rows, and since the product BA exists, B must have m columns so that B is an $n \times m$ matrix. To finish the proof, we need only verify that $m = n$.

We first apply Lemma 2.60 with A and B , to obtain the inequality $m \leq n$. We then apply Lemma 2.60 again (switching the order of the matrices), to obtain the inequality $n \leq m$. It follows that $m = n$, as we wanted. ■

Of course, not all square matrices are invertible. In particular, zero matrices are not invertible, along with many other square matrices.

The following proposition will be useful in proving the next theorem.

Proposition 2.62: Reduced Row-Echelon Form of a Square Matrix

If R is the reduced row-echelon form of a square matrix, then either R has a row of zeros or R is an identity matrix.

The proof of this proposition is left as an exercise to the reader. We now consider the second important theorem of this section.

Theorem 2.63: Unique Inverse of a Matrix

Suppose A and B are square matrices such that $AB = I$ where I is an identity matrix. Then it follows that $BA = I$. Further, both A and B are invertible and $B = A^{-1}$ and $A = B^{-1}$.

Proof: Let R be the reduced row-echelon form of a square matrix A . Then, $R = EA$ where E is an invertible matrix. Since $AB = I$, Lemma 2.59 gives us that R does not have a row of zeros. By noting that R is a square matrix and applying Proposition 2.62, we see that $R = I$. Hence, $EA = I$.

Using both that $EA = I$ and $AB = I$, we can finish the proof with a chain of equalities as given by

$$\begin{aligned} BA = IBIA &= (EA)B(E^{-1}E)A \\ &= E(AB)E^{-1}(EA) \\ &= EIE^{-1}I \\ &= EE^{-1} = I \end{aligned}$$

It follows from the definition of the inverse of a matrix that $B = A^{-1}$ and $A = B^{-1}$. ■

This theorem is very useful, since with it we need only test one of the products AB or BA in order to check that B is the inverse of A . The hypothesis that A and B are square matrices is very important, and without this the theorem does not hold.

We will now consider an example.

Example 2.64: Non Square Matrices

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Show that $A^T A = I$ but $AA^T \neq 0$.

Solution. Consider the product $A^T A$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $A^T A = I_2$, where I_2 is the 2×2 identity matrix. However, the product AA^T is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence AA^T is not the 3×3 identity matrix. This shows that for Theorem 2.63, it is essential that both matrices be square and of the same size. \square

Is it possible to have matrices A and B such that $AB = I$, while $BA = 0$? This question is left to the reader to answer, and you should take a moment to consider the answer.

We conclude this section with an important theorem.

Theorem 2.65: The Reduced Row-Echelon Form of an Invertible Matrix

For any matrix A the following conditions are equivalent:

- A is invertible
- The reduced row-echelon form of A is an identity matrix

Proof. In order to prove this, we show that for any given matrix A , each condition implies the other. We first show that if A is invertible, then its reduced row-echelon form is an identity matrix, then we show that if the reduced row-echelon form of A is an identity matrix, then A is invertible.

If A is invertible, there is some matrix B such that $AB = I$. By Lemma 2.59, we get that the reduced row-echelon form of A does not have a row of zeros. Then by Theorem 2.61, it follows that A and the reduced row-echelon form of A are square matrices. Finally, by Proposition 2.62, this reduced row-echelon form of A must be an identity matrix. This proves the first implication.

Now suppose the reduced row-echelon form of A is an identity matrix I . Then $I = EA$ for some product E of elementary matrices. By Theorem 2.63, we can conclude that A is invertible. \square

Theorem 2.65 corresponds to Algorithm 2.37, which claims that A^{-1} is found by row reducing the augmented matrix $[A|I]$ to the form $[I|A^{-1}]$. This will be a matrix product $E[A|I]$ where E is a product of elementary matrices. By the rules of matrix multiplication, we have that $E[A|I] = [EA|EI] = [EA|E]$.

It follows that the reduced row-echelon form of $[A|I]$ is $[EA|E]$, where EA gives the reduced row-echelon form of A . By Theorem 2.65, if $EA \neq I$, then A is not invertible, and if $EA = I$, A is invertible. If $EA = I$, then by Theorem 2.63, $E = A^{-1}$. This proves that Algorithm 2.37 does in fact find A^{-1} .

2.1.11. EXERCISES

1. For the following pairs of matrices, determine if the sum $A + B$ is defined. If so, find the sum.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 4 \end{bmatrix}$

2. For each matrix A , find the matrix $-A$ such that $A + (-A) = 0$.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

3. In the context of Proposition 2.7, describe $-A$ and 0 .

4. For each matrix A , find the product $(-2)A$, $0A$, and $3A$.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

5. Using only the properties given in Proposition 2.7 and Proposition 2.10, show $-A$ is unique.
6. Using only the properties given in Proposition 2.7 and Proposition 2.10 show 0 is unique.
7. Using only the properties given in Proposition 2.7 and Proposition 2.10 show $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero matrix of appropriate size.
8. Using only the properties given in Proposition 2.7 and Proposition 2.10, as well as previous problems show $(-1)A = -A$.
9. Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$, $E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the following if possible. If it is not possible explain why.

- (a) $-3A$
- (b) $3B - A$
- (c) AC
- (d) CB
- (e) AE
- (f) EA

10. Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

- (a) $-3A$
- (b) $3B - A$
- (c) AC
- (d) CA
- (e) AE
- (f) EA
- (g) BE
- (h) DE

11. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$. Find the following if possible.

- (a) AB
- (b) BA
- (c) AC
- (d) CA
- (e) CB
- (f) BC

12. Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.
13. Let $X = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. Find $X^T Y$ and XY^T if possible.
14. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?
15. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?
16. Find 2×2 matrices, A , B , and C such that $A \neq 0$, $C \neq B$, but $AC = AB$.
17. Give an example of matrices (of any size), A, B, C such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.
18. Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.
19. Give an example of matrices (of any size), A, B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.
20. Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ with $AB \neq BA$.
21. Write the system

$$\begin{aligned} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

22. Write the system

$$\begin{aligned}x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1\end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

23. Write the system

$$\begin{aligned}x_1 + x_2 + x_3 \\ 2x_3 + x_1 + x_2 \\ x_3 - x_1 \\ 3x_4 + x_1\end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

24. A matrix A is called *idempotent* if $A^2 = A$. Let

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

and show that A is idempotent .

25. For each pair of matrices, find the $(1, 2)$ -entry and $(2, 3)$ -entry of the product AB .

$$(a) \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 16 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

26. Suppose A and B are square matrices of the same size. Which of the following are necessarily true?

$$(a) \quad (A - B)^2 = A^2 - 2AB + B^2$$

$$(b) \quad (AB)^2 = A^2B^2$$

$$(c) \quad (A + B)^2 = A^2 + 2AB + B^2$$

$$(d) \quad (A + B)^2 = A^2 + AB + BA + B^2$$

$$(e) \quad A^2B^2 = A(AB)B$$

$$(f) (A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$(g) (A + B)(A - B) = A^2 - B^2$$

27. Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

(a) $-3A^T$

(b) $3B - A^T$

(c) $E^T B$

(d) EE^T

(e) $B^T B$

(f) CA^T

(g) $D^T BE$

28. Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix. **Hint:** Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.

29. Show that the main diagonal of every skew symmetric matrix consists of only zeros. Recall that the main diagonal consists of every entry of the matrix which is of the form a_{ii} .

30. Prove 3. That is, show that for an $m \times n$ matrix A , an $n \times p$ matrix B , and scalars r, s , the following holds:

$$(rA + sB)^T = rA^T + sB^T$$

31. Prove that $I_m A = A$ where A is an $m \times n$ matrix.

32. Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

33. Suppose $AB = AC$ and A is a non invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

34. Give an example of a matrix A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

35. Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

36. Let

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

37. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

38. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

39. Let A be a 2×2 invertible matrix, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a formula for A^{-1} in terms of a, b, c, d .

40. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

41. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

42. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

43. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

44. Using the inverse of the matrix, find the solution to the systems:

(a)

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now give the solution in terms of a and b to

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

45. Using the inverse of the matrix, find the solution to the systems:

(a)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now give the solution in terms of a, b , and c to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

46. Show that if A is an $n \times n$ invertible matrix and X is a $n \times 1$ matrix such that $AX = B$ for B an $n \times 1$ matrix, then $X = A^{-1}B$.

47. Prove that if A^{-1} exists and $AX = 0$ then $X = 0$.

48. Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

49. Show that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

50. Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB(B^{-1}A^{-1}) = I$$

and

$$B^{-1}A^{-1}(AB) = I$$

Hint: Use Problem 48.

51. Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that $(ABC)(C^{-1}B^{-1}A^{-1}) = I$ and $(C^{-1}B^{-1}A^{-1})(ABC) = I$. **Hint:** Use Problem 48.
52. If A is invertible, show $(A^2)^{-1} = (A^{-1})^2$. **Hint:** Use Problem 48.
53. If A is invertible, show $(A^{-1})^{-1} = A$. **Hint:** Use Problem 48.
54. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.
55. Let $A = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.
56. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.
57. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 5 & 1 \end{bmatrix}$.
- Find the elementary matrix E such that $EA = B$.
 - Find the inverse of E , E^{-1} , such that $E^{-1}B = A$.
58. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}$.
- Find the elementary matrix E such that $EA = B$.
 - Find the inverse of E , E^{-1} , such that $E^{-1}B = A$.
59. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}$.

- (a) Find the elementary matrix E such that $EA = B$.
- (b) Find the inverse of E , E^{-1} , such that $E^{-1}B = A$.

60. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}.$$

- (a) Find the elementary matrix E such that $EA = B$.
- (b) Find the inverse of E , E^{-1} , such that $E^{-1}B = A$.

3. DETERMINANTS

3.1 BASIC TECHNIQUES AND PROPERTIES

Outcomes

- A. Evaluate the determinant of a square matrix using either Laplace Expansion or row operations.
- B. Demonstrate the effects that row operations have on determinants.
- C. Verify the following:
 - (a) The determinant of a product of matrices is the product of the determinants.
 - (b) The determinant of a matrix is equal to the determinant of its transpose.

3.1.1. COFACTORS AND 2×2 DETERMINANTS

Let A be an $n \times n$ matrix. That is, let A be a square matrix. The **determinant** of A , denoted by $\det(A)$ is a very important number which we will explore throughout this section.

If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 3.1: Determinant of a Two By Two Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A) = ad - cb$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following is an example of finding the determinant of a 2×2 matrix.

Example 3.2: A Two by Two Determinant

Find $\det(A)$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

Solution. From Definition 3.1,

$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

□

The 2×2 determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools including the 2×2 determinant.

We begin with the following definition.

Definition 3.3: The ij^{th} Minor of a Matrix

Let A be a 3×3 matrix. The ij^{th} **minor** of A , denoted as $\text{minor}(A)_{ij}$, is the determinant of the 2×2 matrix which results from deleting the i^{th} row and the j^{th} column of A .

In general, if A is an $n \times n$ matrix, then the ij^{th} minor of A is the determinant of the $n - 1 \times n - 1$ matrix which results from deleting the i^{th} row and the j^{th} column of A .

Hence, there is a minor associated with each entry of A . Consider the following example which demonstrates this definition.

Example 3.4: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\text{minor}(A)_{12}$ and $\text{minor}(A)_{23}$.

Solution. First we will find $\text{minor}(A)_{12}$. By Definition 3.3, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$\text{minor}(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Using Definition 3.1, we see that

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (4)(1) - (3)(2) = 4 - 6 = -2$$

Therefore $\text{minor}(A)_{12} = -2$.

Similarly, $\text{minor}(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\text{minor}(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of A is left as an exercise. \square

The ij^{th} minor of a matrix A is used in another important definition, given next.

Definition 3.5: The ij^{th} Cofactor of a Matrix

Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, denoted by $\text{cof}(A)_{ij}$ is defined to be

$$\text{cof}(A)_{ij} = (-1)^{i+j} \text{minor}(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If a_{ij} is the ij^{th} entry of the matrix, then its cofactor is just $\text{cof}(A)_{ij}$.

Example 3.6: Finding Cofactors of a Matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\text{cof}(A)_{12}$ and $\text{cof}(A)_{23}$.

Solution. We will use Definition 3.5 to compute these cofactors.

First, we will compute $\text{cof}(A)_{12}$. Therefore, we need to find $\text{minor}(A)_{12}$. This is the determinant of the 2×2 matrix which results when you delete the first row and the second column. Thus $\text{minor}(A)_{12}$ is given by

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

Then,

$$\text{cof}(A)_{12} = (-1)^{1+2} \text{minor}(A)_{12} = (-1)^{1+2} (-2) = 2$$

Hence, $\text{cof}(A)_{12} = 2$.

Similarly, we can find $\text{cof}(A)_{23}$. First, find $\text{minor}(A)_{23}$, which is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Hence,

$$\text{cof}(A)_{23} = (-1)^{2+3} \text{minor}(A)_{23} = (-1)^{2+3} (-4) = 4$$

□

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of a 3×3 matrix.

Definition 3.7: The Determinant of a Three By Three Matrix

Let A be a 3×3 matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.

This process when applied to the i^{th} row (column) is known as **expanding along the i^{th} row (column)** as is given by

$$\det(A) = a_{i1}\text{cof}(A)_{i1} + a_{i2}\text{cof}(A)_{i2} + a_{i3}\text{cof}(A)_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix A . This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

Example 3.8: Finding the Determinant of a Three by Three Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Laplace Expansion.

Solution. First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 3.7, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = \overbrace{1(-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{11}} + \overbrace{4(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 3.7, we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = \overbrace{4(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\text{cof}(A)_{22}} + \overbrace{2(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$. \square

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

We present this idea formally in the following theorem.

Theorem 3.9: The Determinant is Well Defined

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

We have now looked at the determinant of 2×2 and 3×3 matrices. It turns out that the method used to calculate the determinant of a 3×3 matrix can be used to calculate the determinant of any sized matrix. Notice that Definition 3.3, Definition 3.5 and Definition 3.7 can all be applied to a matrix of any size.

For example, the ij^{th} minor of a 4×4 matrix is the determinant of the 3×3 matrix you obtain when you delete the i^{th} row and the j^{th} column. Just as with the 3×3 determinant, we can compute the determinant of a 4×4 matrix by Laplace Expansion, along any row or column

Consider the following example.

Example 3.10: Determinant of a Four by Four Matrix

Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

Solution. As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. Then, using Laplace Expansion,

$$\det(A) = 3(-1)^{1+3} \begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3} \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$

$$4(-1)^{3+3} \begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3} \begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each 3×3 determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$. \square

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 3.11: The Determinant of an $n \times n$ Matrix

Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^n a_{ij} \text{cof}(A)_{ij} = \sum_{i=1}^n a_{ij} \text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

In the following sections, we will explore some important properties and characteristics of the determinant.

3.1.2. THE DETERMINANT OF A TRIANGULAR MATRIX

There is a certain type of matrix for which finding the determinant is a very simple procedure. Consider the following definition.

Definition 3.12: Triangular Matrices

A matrix A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries of such a matrix below the main diagonal equal 0, as shown. Here, $*$ refers to any nonzero number.

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

Theorem 3.13: Determinant of a Triangular Matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Consider the following example.

Example 3.14: Determinant of a Triangular Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

Solution. From Theorem 3.13, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = 1 \times 2 \times 3 \times (-1) = -6$.

Without using Theorem 3.13, you could use Laplace Expansion. We will expand along the first column. This gives

$$\begin{aligned} \det(A) = & 1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + \\ & 0(-1)^{3+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix} \end{aligned}$$

and the only nonzero term in the expansion is

$$1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$\begin{aligned} \det(A) &= 1 \times \left(2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix} \right) \\ &= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} \end{aligned}$$

Next use Definition 3.1 to find the determinant of this 2×2 matrix, which is just $3 \times -1 - 0 \times 33.7 = -3$. Putting all these steps together, we have

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix! \square

You can see that while both methods result in the same answer, Theorem 3.13 provides a much quicker method.

In the next section, we explore some important properties of determinants.

3.1.3. PROPERTIES OF DETERMINANTS

There are many important properties of determinants. Since many of these properties involve the row operations discussed in Chapter 1, we recall that definition now.

Definition 3.15: Row Operations

The row operations consist of the following

1. *Switch two rows.*
2. *Multiply a row by a nonzero number.*
3. *Replace a row by a multiple of another row added to itself.*

We will now consider the effect of row operations on the determinant of a matrix. In future sections, we will see that using the following properties can greatly assist in finding determinants.

The first theorem explains the affect on the determinant of a matrix when two rows are switched.

Theorem 3.16: Switching Rows

Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows of A . Then $\det(B) = -\det(A)$.

When we switch two rows of a matrix, the determinant is multiplied by -1 . Consider the following example.

Example 3.17: Switching Two Rows

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution. By Definition 3.1, $\det(A) = 1 \times 4 - 3 \times 2 = -2$. Notice that the rows of B are the rows of A but switched. By Theorem 3.16 since two rows of A have been switched, $\det(B) = -\det(A) = -(-2) = 2$. You can verify this using Definition 3.1. \square

The next theorem demonstrates the effect on the determinant of a matrix when we multiply a row by a scalar.

Theorem 3.18: Multiplying a Row by a Scalar

Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k . Then $\det(B) = k \det(A)$.

Notice that this theorem is true when we multiply *one* row of the matrix by k . If we were to multiply *two* rows of A by k to obtain B , we would have $\det(B) = k^2 \det(A)$. Suppose we were to multiply all n rows of A by k to obtain the matrix B , so that $B = kA$. Then, $\det(B) = k^n \det(A)$. This gives the next theorem.

Theorem 3.19: Scalar Multiplication

Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Consider the following example.

Example 3.20: Multiplying a Row by 5

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution. By Definition 3.1, $\det(A) = -2$. We can also compute $\det(B)$ using Definition 3.1, and we see that $\det(B) = -10$.

Now, let's compute $\det(B)$ using Theorem 3.18 and see if we obtain the same answer. Notice that the first row of B is 5 times the first row of A , while the second row of B is equal to the second row of A . By Theorem 3.18, $\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

You can see that this matches our answer above. \square

Finally, consider the next theorem for the last row operation, that of adding a multiple of a row to another row.

Theorem 3.21: Adding a Multiple of a Row to Another Row

Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Therefore, when we add a multiple of a row to another row, the determinant of the matrix is unchanged. Note that if a matrix A contains a row which is a multiple of another row, $\det(A)$ will equal 0. To see this, suppose the first row of A is equal to -1 times the second row. By Theorem 3.21, we can add the first row to the second row, and the determinant will be unchanged. However, this row operation will result in a row of zeros. Using Laplace Expansion along the row of zeros, we find that the determinant is 0.

Consider the following example.

Example 3.22: Adding a Row to Another Row

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. Find $\det(B)$.

Solution. By Definition 3.1, $\det(A) = -2$. Notice that the second row of B is two times the first row of A added to the second row. By Theorem 3.16, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.1. \square

Example 3.23: Multiple of a Row

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Show that $\det(A) = 0$.

Solution. Using Definition 3.1, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

However notice that the second row is equal to 2 times the first row. Then by the discussion above following Theorem 3.21 the determinant will equal 0. \square

Until now, our focus has primarily been on row operations. However, we can carry out the same operations with columns, rather than rows. The three operations outlined in Definition 3.15 can be done with columns instead of rows. In this case, in Theorems 3.16, 3.18, and 3.21 you can replace the word, "row" with the word "column".

There are several other major properties of determinants which do not involve row (or column) operations. The first is the determinant of a product of matrices.

Theorem 3.24: Determinant of a Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B)$$

In order to find the determinant of a product of matrices, we can simply take the product of the determinants.

Consider the following example.

Example 3.25: The Determinant of a Product

Compare $\det(AB)$ and $\det(A) \det(B)$ for

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Solution. First compute AB , which is given by

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

and so by Definition 3.1

$$\det(AB) = \det \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix} = -40$$

Now

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = 8$$

and

$$\det(B) = \det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5$$

Computing $\det(A) \times \det(B)$ we have $8 \times -5 = -40$. This is the same answer as above and you can see that $\det(A) \det(B) = 8 \times (-5) = -40 = \det(AB)$. \square

Consider the next important property.

Theorem 3.26: Determinant of the Transpose

Let A be a matrix where A^T is the transpose of A . Then,

$$\det(A^T) = \det(A)$$

This theorem is illustrated in the following example.

Example 3.27: Determinant of the Transpose

Let

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

Find $\det(A^T)$.

Solution. First, note that

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

Using Definition 3.1, we can compute $\det(A)$ and $\det(A^T)$. It follows that $\det(A) = 2 \times 3 - 4 \times 5 = -14$ and $\det(A^T) = 2 \times 3 - 5 \times 4 = -14$. Hence, $\det(A) = \det(A^T)$. \square

The following provides an essential property of the determinant, as well as a useful way to determine if a matrix is invertible.

Theorem 3.28: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Consider the following example.

Example 3.29: Determinant of an Invertible Matrix

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. For each matrix, determine if it is invertible. If so, find the determinant of the inverse.

Solution. Consider the matrix A first. Using Definition 3.1 we can find the determinant as follows:

$$\det(A) = 3 \times 4 - 2 \times 6 = 12 - 12 = 0$$

By Theorem 3.28 A is not invertible.

Now consider the matrix B . Again by Definition 3.1 we have

$$\det(B) = 2 \times 1 - 5 \times 3 = 2 - 15 = -13$$

By Theorem 3.28 B is invertible and the determinant of the inverse is given by

$$\begin{aligned} \det(A^{-1}) &= \frac{1}{\det(A)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13} \end{aligned}$$

3.1.4. FINDING DETERMINANTS USING ROW OPERATIONS

Theorems 3.16, 3.18 and 3.21 illustrate how row operations affect the determinant of a matrix. In this section, we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but timely, as there are many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

Example 3.30: Finding a Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

Solution. We will use the properties of determinants outlined above to find $\det(A)$. First, add -5 times the first row to the second row. Then add -4 times the first row to the third row, and -2 times the first row to the fourth row. This yields the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix}$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 3.21, $\det(B) = \det(A)$.

At this stage, you could use Laplace Expansion to find $\det(B)$. However, we will continue with row operations to find an even simpler matrix to work with.

Add -3 times the third row to the second row. By Theorem 3.21 this does not change the value of the determinant. Then, multiply the fourth row by -3 . This results in the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

Here, $\det(C) = -3 \det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right) \det(C)$

Since $\det(A) = \det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right) \det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Now replace the add 2 times the third row to the fourth row. This does not change the value of the determinant by Theorem 3.21. Finally switch the third and second rows. This causes the determinant to be multiplied by -1 . Thus $\det(C) = -\det(D)$ where

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix}$$

$$\text{Hence, } \det(A) = \left(-\frac{1}{3}\right) \det(C) = \left(\frac{1}{3}\right) \det(D)$$

You could do more row operations or you could note that this can be easily expanded along the first column. Then, expand the resulting 3×3 matrix also along the first column. This results in

$$\det(D) = 1(-3) \begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right) (1485) = 495$. □

You can see that by using row operations, we can simplify a matrix to the point where Laplace Expansion involves only a few steps. In Example 3.30, we also could have continued until the matrix was in upper triangular form, and taken the product of the entries on the main diagonal. Whenever computing the determinant, it is useful to consider all the possible methods and tools.

Consider the next example.

Example 3.31: Find the Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

Solution. Once again, we will simplify the matrix through row operations. Add -1 times the first row to the second row. Next add -2 times the first row to the third and finally take -3 times the first row and add to the fourth row. This yields

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.21, $\det(A) = \det(B)$.

Remember you can work with the columns also. Take -5 times the fourth column and add to the second column. This yields

$$C = \begin{bmatrix} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.21 $\det(A) = \det(C)$.

Now take -1 times the third row and add to the top row. This gives.

$$D = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

which by Theorem 3.21 has the same determinant as A .

Now, we can find $\det(D)$ by expanding along the first column as follows. You can see that there will be only one non zero term.

$$\det(D) = 1 \det \begin{bmatrix} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{bmatrix} + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1 \left(0 + 8 \det \begin{bmatrix} -1 & -1 \\ -8 & -4 \end{bmatrix} + 10 \det \begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix} \right) = -82$$

Now since $\det(A) = \det(D)$, it follows that $\det(A) = -82$. \square

Remember that you can verify these answers by using Laplace Expansion on A . Similarly, if you first compute the determinant using Laplace Expansion, you can use the row operation method to verify.

3.1.5. EXERCISES

- Find the determinants of the following matrices.

(a) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. Find the following.

(a) $\text{minor}(A)_{11}$

(b) $\text{minor}(A)_{21}$

(c) $\text{minor}(A)_{32}$

(d) $\text{cof}(A)_{11}$

(e) $\text{cof}(A)_{21}$

(f) $\text{cof}(A)_{32}$

3. Find the determinants of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

4. Find the following determinant by expanding along the first row and second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

5. Find the following determinant by expanding along the first column and third row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

6. Find the following determinant by expanding along the second row and first column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

7. Compute the determinant by cofactor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

8. Find the determinant of the following matrices.

(a) $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

9. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

10. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

11. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

12. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

13. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

14. Let A be an $r \times r$ matrix and suppose there are $r - 1$ rows (columns) such that all rows (columns) are linear combinations of these $r - 1$ rows (columns). Show $\det(A) = 0$.

15. Show $\det(aA) = a^n \det(A)$ for an $n \times n$ matrix A and scalar a .

16. Construct 2×2 matrices A and B to show that the $\det A \det B = \det(AB)$.
17. Is it true that $\det(A + B) = \det(A) + \det(B)$? If this is so, explain why. If it is not so, give a counter example.
18. An $n \times n$ matrix is called **nilpotent** if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?
19. A matrix is said to be **orthogonal** if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?
20. Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then $\det(A) = \det(B)$.
21. Tell whether each statement is true or false. If true, provide a proof. If false, provide a counter example.
- (a) If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.
 - (b) If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
 - (c) For two $n \times n$ matrices A and B , $\det(A + B) = \det(A) + \det(B)$.
 - (d) For an $n \times n$ matrix A , $\det(3A) = 3 \det(A)$
 - (e) If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.
 - (f) If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4 \det(A)$.
 - (g) For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$.
 - (h) If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.
 - (i) If $A^k = 0$ for some positive integer k , then $\det(A) = 0$.
 - (j) If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.
22. Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

23. Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

24. Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

25. Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix}$$

3.2 APPLICATIONS OF THE DETERMINANT

Outcomes

- A. Use determinants to determine whether a matrix has an inverse, and evaluate the inverse using cofactors.
- B. Apply Cramer's Rule to solve a 2×2 or a 3×3 linear system.
- C. Given data points, find an appropriate interpolating polynomial and use it to estimate points.

3.2.1. A FORMULA FOR THE INVERSE

The determinant of a matrix also provides a way to find the inverse of a matrix. Recall the definition of the inverse of a matrix in Definition 2.33. We say that A^{-1} , an $n \times n$ matrix, is the inverse of A , also $n \times n$, if $AA^{-1} = I$ and $A^{-1}A = I$.

We now define a new matrix called the **cofactor matrix** of A . The cofactor matrix of A is the matrix whose ij^{th} entry is the ij^{th} cofactor of A . The formal definition is as follows.

Definition 3.32: The Cofactor Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the **cofactor matrix** of A , denoted $\text{cof}(A)$, is defined by $\text{cof}(A) = [\text{cof}(A)_{ij}]$ where $\text{cof}(A)_{ij}$ is the ij^{th} cofactor of A .

Note that $\text{cof}(A)_{ij}$ denotes the ij^{th} entry of the cofactor matrix.

We will use the cofactor matrix to create a formula for the inverse of A . First, we define the **adjugate** of A to be the transpose of the cofactor matrix. We can also call this matrix the **classical adjoint** of A , and we denote it by $\text{adj}(A)$.

In the specific case where A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $\text{adj}(A)$ is given by

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general, $\text{adj}(A)$ can always be found by taking the transpose of the cofactor matrix of A . The following theorem provides a formula for A^{-1} using the determinant and adjugate of A .

Theorem 3.33: The Inverse and the Determinant

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.

If $\det(A) \neq 0$ (so A is invertible), then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

The proof of this Theorem is below, after two examples demonstrating this concept. Notice that this formula is only defined when $\det(A) \neq 0$.

Consider the following example.

Example 3.34: Find Inverse Using the Determinant

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

using the formula in Theorem 3.33.

Solution. According to Theorem 3.33,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

First we will find the determinant of this matrix. Using Theorems 3.16, 3.18, and 3.21, we can first simplify the matrix through row operations. First, add -3 times the first row to the second row. Then add -1 times the first row to the third row to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem 3.21, $\det(A) = \det(B)$. By Theorem 3.13, $\det(B) = 1 \times -6 \times -2 = 12$. Hence, $\det(A) = 12$.

Now, we need to find $\text{adj}(A)$. To do so, first we will find the cofactor matrix of A . This is given by

$$\text{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}$$

Here, the ij^{th} entry is the ij^{th} cofactor of the original matrix A which you can verify. Therefore, from Theorem 3.33, the inverse of A is given by

$$A^{-1} = \frac{1}{12} \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Remember that we can always verify our answer for A^{-1} . Compute the product AA^{-1} and $A^{-1}A$ and make sure each product is equal to I .

Compute $A^{-1}A$ as follows

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

You can verify that $AA^{-1} = I$ and hence our answer is correct. \square

We will look at another example of how to use this formula to find A^{-1} .

Example 3.35: Find the Inverse From a Formula

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

using the formula given in Theorem 3.33.

Solution. First we need to find $\det(A)$. This step is left as an exercise and you should verify that $\det(A) = \frac{1}{6}$. The inverse is therefore equal to

$$A^{-1} = \frac{1}{(1/6)} \text{adj}(A) = 6 \text{adj}(A)$$

We continue to calculate as follows. Here we show the 2×2 determinants needed to find the cofactors.

$$A^{-1} = 6 \begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6 \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Again, you can always check your work by multiplying $A^{-1}A$ and AA^{-1} and ensuring these products equal I .

$$A^{-1}A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This tells us that our calculation for A^{-1} is correct. It is left to the reader to verify that $AA^{-1} = I$. \square

The verification step is very important, as it is a simple way to check your work! If you multiply $A^{-1}A$ and AA^{-1} and these products are not both equal to I , be sure to go back and double check each step. One common error is to forget to take the transpose of the cofactor matrix, so be sure to complete this step.

We will now prove Theorem 3.33.

Proof. First if A is invertible, then by Theorem ?? we have:

$$1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1})$$

and thus $\det(A) \neq 0$.

Equivalently, if $\det(A) = 0$, then A is not invertible.

Now assume $\det(A) \neq 0$. From the definition of the determinant in terms of expansion along a column, and letting $A = [a_{ir}]$, if $\det(A) \neq 0$,

$$\sum_{i=1}^n a_{ir} \text{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1$$

Now consider

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix B_k whose determinant equals zero by Theorem 3.16. However, expanding this matrix B_k along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk} = \begin{cases} 1 & \text{if } r = k \\ 0 & \text{if } r \neq k \end{cases}$$

Now

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} = \sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ki}^T$$

which is the kr^{th} entry of $\operatorname{cof}(A)^T A$. Therefore,

$$\frac{\operatorname{cof}(A)^T}{\det(A)} A = I \quad (3.1)$$

Using the other formula in Definition 3.11, and similar reasoning,

$$\sum_{j=1}^n a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

Now

$$\sum_{j=1}^n a_{rj} \operatorname{cof}(A)_{kj} = \sum_{j=1}^n a_{rj} \operatorname{cof}(A)_{jk}^T$$

which is the rk^{th} entry of $A \operatorname{cof}(A)^T$. Therefore,

$$A \frac{\operatorname{cof}(A)^T}{\det(A)} = I \quad (3.2)$$

and it follows from 3.1 and 3.2 that $A^{-1} = [a_{ij}]^{-1}$, where

$$[a_{ij}]^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}$$

In other words,

$$A^{-1} = \frac{\operatorname{cof}(A)^T}{\det(A)}$$

□

This method for finding the inverse of A is useful in many contexts. In particular, it is useful with complicated matrices where the entries are functions, rather than numbers.

Consider the following example.

Example 3.36: Inverse for Non-Constant Matrix

Suppose

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

Show that $A(t)^{-1}$ exists and then find it.

Solution. First note $\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t \neq 0$ so $A(t)^{-1}$ exists.

The cofactor matrix is

$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}$$

and so the inverse is

$$\frac{1}{e^t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}^T = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

□

3.2.2. CRAMER'S RULE

Another context in which the formula given in Theorem 3.33 is important is **Cramer's Rule**. Recall that we can represent a system of linear equations in the form $AX = B$, where the solutions to this system are given by X . Cramer's Rule gives a formula for the solutions X in the special case that A is a square invertible matrix. Note this rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Suppose we have a system of equations given by $AX = B$, and we want to find solutions X which satisfy this system. Then recall that if A^{-1} exists,

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Hence, the solutions X to the system are given by $X = A^{-1}B$. Since we assume that A^{-1} exists, we can use the formula for A^{-1} given above. Substituting this formula into the equation for X , we have

$$X = A^{-1}B = \frac{1}{\det(A)} \text{adj}(A) B$$

Let x_i be the i^{th} entry of X and b_j be the j^{th} entry of B . Then this equation becomes

$$x_i = \sum_{j=1}^n [a_{ij}]^{-1} b_j = \sum_{j=1}^n \frac{1}{\det(A)} \text{adj}(A)_{ij} b_j$$

where $\text{adj}(A)_{ij}$ is the ij^{th} entry of $\text{adj}(A)$.

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{bmatrix} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{bmatrix}$$

where here the i^{th} column of A is replaced with the column vector $[b_1, \dots, b_n]^T$. The determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

We formally define this method now.

Procedure 3.37: Using Cramer's Rule

Suppose A is an $n \times n$ invertible matrix and we wish to solve the system $AX = B$ for $X = [x_1, \dots, x_n]^T$. Then Cramer's rule says

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the i^{th} column of A with the column matrix

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

We illustrate this procedure in the following example.

Example 3.38: Using Cramer's Rule

Find x, y, z if

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution. We will use method outlined in Procedure 3.37 to find the values for x, y, z which give the solution to this system. Let

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In order to find x , we calculate

$$x = \frac{\det(A_1)}{\det(A)}$$

where A_1 is the matrix obtained from replacing the first column of A with B .

Hence, A_1 is given by

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{bmatrix}$$

Therefore,

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{1}{2}$$

Similarly, to find y we construct A_2 by replacing the second column of A with B . Hence, A_2 is given by

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore,

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = -\frac{1}{7}$$

Similarly, A_3 is constructed by replacing the third column of A with B . Then, A_3 is given by

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{bmatrix}$$

Therefore, z is calculated as follows.

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{11}{14}$$

□

Cramer's Rule gives you another tool to consider when solving a system of linear equations.

We can also use Cramer's Rule for systems of non linear equations. Consider the following system where the matrix A has functions rather than numbers for entries.

Example 3.39: Use Cramer's Rule for Non-Constant Matrix

Solve for z if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

Solution. We are asked to find the value of z in the solution. We will solve using Cramer's rule. Thus

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = t((\cos t)t + \sin t)e^{-t}$$

□

3.2.3. POLYNOMIAL INTERPOLATION

In studying a set of data that relates variables x and y , it may be the case that we can use a polynomial to “fit” to the data. If such a polynomial can be established, it can be used to estimate values of x and y which have not been provided.

Consider the following example.

Example 3.40: Polynomial Interpolation

Given data points $(1, 4)$, $(2, 9)$, $(3, 12)$, find an interpolating polynomial $p(x)$ of degree at most 2 and then estimate the value corresponding to $x = \frac{1}{2}$.

Solution. We want to find a polynomial given by

$$p(x) = r_0 + r_1x_1 + r_2x_2^2$$

such that $p(1) = 4$, $p(2) = 9$ and $p(3) = 12$. To find this polynomial, substitute the known values in for x and solve for r_0 , r_1 , and r_2 .

$$\begin{aligned} p(1) &= r_0 + r_1 + r_2 = 4 \\ p(2) &= r_0 + 2r_1 + 4r_2 = 9 \\ p(3) &= r_0 + 3r_1 + 9r_2 = 12 \end{aligned}$$

Writing the augmented matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 12 \end{array} \right]$$

After row operations, the resulting matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Therefore the solution to the system is $r_0 = -3$, $r_1 = 8$, $r_2 = -1$ and the required interpolating polynomial is

$$p(x) = -3 + 8x - x^2$$

To estimate the value for $x = \frac{1}{2}$, we calculate $p(\frac{1}{2})$:

$$\begin{aligned} p\left(\frac{1}{2}\right) &= -3 + 8\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 \\ &= -3 + 4 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

□

This procedure can be used for any number of data points, and any degree of polynomial. The steps are outlined below.

Procedure 3.41: Finding an Interpolating Polynomial

Suppose that values of x and corresponding values of y are given, such that the actual relationship between x and y is unknown. Then, values of y can be estimated using an **interpolating polynomial** $p(x)$. If given x_1, \dots, x_n and the corresponding y_1, \dots, y_n , the procedure to find $p(x)$ is as follows:

1. The desired polynomial $p(x)$ is given by

$$p(x) = r_0 + r_1x + r_2x^2 + \dots + r_{n-1}x^{n-1}$$

2. $p(x_i) = y_i$ for all $i = 1, 2, \dots, n$ so that

$$\begin{aligned} r_0 + r_1x_1 + r_2x_1^2 + \dots + r_{n-1}x_1^{n-1} &= y_1 \\ r_0 + r_1x_2 + r_2x_2^2 + \dots + r_{n-1}x_2^{n-1} &= y_2 \\ &\vdots \\ r_0 + r_1x_n + r_2x_n^2 + \dots + r_{n-1}x_n^{n-1} &= y_n \end{aligned}$$

3. Set up the augmented matrix of this system of equations

$$\left[\begin{array}{cccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array} \right]$$

4. Solving this system will result in a unique solution r_0, r_1, \dots, r_{n-1} . Use these values to construct $p(x)$, and estimate the value of $p(a)$ for any $x = a$.

This procedure motivates the following theorem.

Theorem 3.42: Polynomial Interpolation

Given n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial $p(x) = r_0 + r_1x + r_2x^2 + \dots + r_{n-1}x^{n-1}$ such that $p(x_i) = y_i$ for $i = 1, 2, \dots, n$. The resulting polynomial $p(x)$ is called the **interpolating polynomial** for the data points.

We conclude this section with another example.

Example 3.43: Polynomial Interpolation

Consider the data points $(0, 1), (1, 2), (3, 22), (5, 66)$. Find an interpolating polynomial $p(x)$ of degree at most three, and estimate the value of $p(2)$.

Solution. The desired polynomial $p(x)$ is given by:

$$p(x) = r_0 + r_1x + r_2x^2 + r_3x^3$$

Using the given points, the system of equations is

$$\begin{aligned} p(0) &= r_0 = 1 \\ p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\ p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 22 \\ p(5) &= r_0 + 5r_1 + 25r_2 + 125r_3 = 66 \end{aligned}$$

The augmented matrix is given by:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 9 & 27 & 22 \\ 1 & 5 & 25 & 125 & 66 \end{array} \right]$$

The resulting matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, $r_0 = 1, r_1 = -2, r_2 = 3, r_3 = 0$ and $p(x) = 1 - 2x + 3x^2$. To estimate the value of $p(2)$, we compute $p(2) = 1 - 2(2) + 3(2^2) = 1 - 4 + 12 = 9$. \square

3.2.4. EXERCISES

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

3. Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

6. For the following matrices, determine if they are invertible. If so, use the formula for the inverse in terms of the cofactor matrix to find each inverse. If the inverse does not exist, explain why.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

8. Consider the matrix

$$A = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

9. Consider the matrix

$$A = \begin{bmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

10. Consider the matrix

$$A = \begin{bmatrix} e^t & e^{-t} \cos t & e^{-t} \sin t \\ e^t & -e^{-t} \cos t - e^{-t} \sin t & -e^{-t} \sin t + e^{-t} \cos t \\ e^t & 2e^{-t} \sin t & -2e^{-t} \cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

11. Show that if $\det(A) \neq 0$ for A an $n \times n$ matrix, it follows that if $AX = 0$, then $X = 0$.
12. Suppose A, B are $n \times n$ matrices and that $AB = I$. Show that then $BA = I$. **Hint:** First explain why $\det(A), \det(B)$ are both nonzero. Then $(AB)A = A$ and then show $BA(BA - I) = 0$. From this use what is given to conclude $A(BA - I) = 0$. Then use Problem 11.
13. Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix

$$A = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & e^t \cos t - e^t \sin t & e^t \cos t + e^t \sin t \end{bmatrix}$$

14. Find the inverse, if it exists, of the matrix

$$A = \begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$$

15. Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Could the same be concluded for lower triangular matrices?
16. If A, B , and C are each $n \times n$ matrices and ABC is invertible, show why each of A, B , and C are invertible.
17. Decide if this statement is true or false: *Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.*

18. Use Cramer's rule to find the solution to

$$\begin{aligned}x + 2y &= 1 \\ 2x - y &= 2\end{aligned}$$

19. Use Cramer's rule to find the solution to

$$\begin{aligned}x + 2y + z &= 1 \\ 2x - y - z &= 2 \\ x + z &= 1\end{aligned}$$

4. \mathbb{R}^n

4.1 VECTORS IN \mathbb{R}^n

Outcomes

A. Find the position vector of a point in \mathbb{R}^n .

The notation \mathbb{R}^n refers to the collection of ordered lists of n real numbers, that is

$$\mathbb{R}^n = \{(x_1 \cdots x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \dots, n\}$$

In this chapter, we take a closer look at vectors in \mathbb{R}^n . First, we will consider what \mathbb{R}^n looks like in more detail. Recall that the point given by $0 = (0, \dots, 0)$ is called the **origin**.

Now, consider the case of \mathbb{R}^n for $n = 1$. Then from the definition we can identify \mathbb{R} with points in \mathbb{R}^1 as follows:

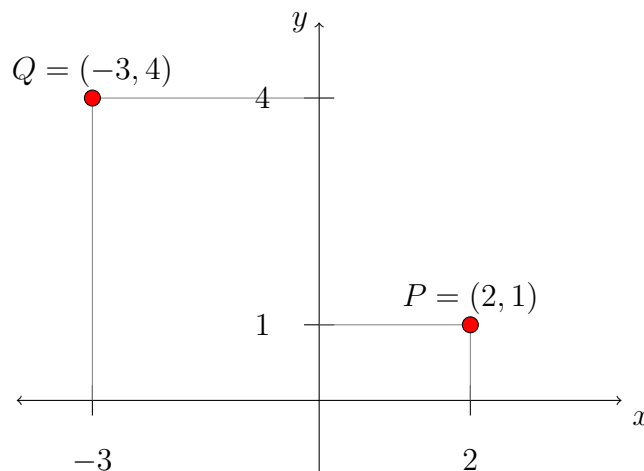
$$\mathbb{R} = \mathbb{R}^1 = \{(x_1) : x_1 \in \mathbb{R}\}$$

Hence, \mathbb{R} is defined as the set of all real numbers and geometrically, we can describe this as all the points on a line.

Now suppose $n = 2$. Then, from the definition,

$$\mathbb{R}^2 = \{(x_1, x_2) : x_j \in \mathbb{R} \text{ for } j = 1, 2\}$$

Consider the familiar coordinate plane, with an x axis and a y axis. Any point within this coordinate plane is identified by where it is located along the x axis, and also where it is located along the y axis. Consider as an example the following diagram.



Hence, every element in \mathbb{R}^2 is identified by two components, x and y , in the usual manner. The coordinates x, y (or x_1, x_2) uniquely determine a point in the plane. Note that while the definition uses x_1 and x_2 to label the coordinates and you may be used to x and y , these notations are equivalent.

Now suppose $n = 3$. You may have previously encountered the 3-dimensional coordinate system, given by

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R} \text{ for } j = 1, 2, 3\}$$

Points in \mathbb{R}^3 will be determined by three coordinates, often written (x, y, z) which correspond to the x, y , and z axes. We can think as above that the first two coordinates determine a point in a plane. The third component determines the height above or below the plane, depending on whether this number is positive or negative, and all together this determines a point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

The idea behind the more general \mathbb{R}^n is that we can extend these ideas beyond $n = 3$. This discussion regarding points in \mathbb{R}^n leads into a study of vectors in \mathbb{R}^n . While we consider \mathbb{R}^n for all n , we will largely focus on $n = 2, 3$ in this section.

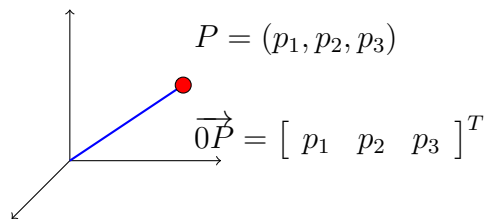
Consider the following definition.

Definition 4.1: The Position Vector

Let $P = (p_1, \dots, p_n)$ be the coordinates of a point in \mathbb{R}^n . Then the vector $\overrightarrow{0P}$ with its tail at $0 = (0, \dots, 0)$ and its tip at P is called the **position vector** of the point P . We write

$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

For this reason we may write both $P = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\overrightarrow{0P} = [p_1 \cdots p_n]^T \in \mathbb{R}^n$. This definition is illustrated in the following picture for the special case of \mathbb{R}^3 .

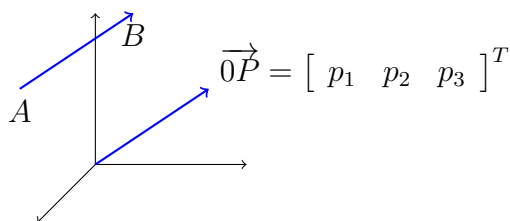


Thus every point P in \mathbb{R}^n determines its position vector $\vec{0P}$. Conversely, every such position vector $\vec{0P}$ which has its tail at 0 and point at P determines the point P of \mathbb{R}^n .

Now suppose we are given two points, P, Q whose coordinates are (p_1, \dots, p_n) and (q_1, \dots, q_n) respectively. We can also determine the **position vector from P to Q** (also called the **vector from P to Q**) defined as follows.

$$\vec{PQ} = \begin{bmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{bmatrix} = \vec{0Q} - \vec{0P}$$

Now, imagine taking a vector in \mathbb{R}^n and moving it around, always keeping it pointing in the same direction as shown in the following picture.



After moving it around, it is regarded as the same vector. Each vector, $\vec{0P}$ and \vec{AB} has the same length (or magnitude) and direction. Therefore, they are equal.

Consider now the general definition for a vector in \mathbb{R}^n .

Definition 4.2: Vectors in \mathbb{R}^n

Let $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \dots, n\}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **vector**. Vectors have both size (magnitude) and direction. The numbers x_j are called the **components** of \vec{x} .

Using this notation, we may use \vec{p} to denote the position vector of point P . Notice that in this context, $\vec{p} = \vec{0P}$. These notations may be used interchangeably.

You can think of the components of a vector as directions for obtaining the vector. Consider $n = 3$. Draw a vector with its tail at the point $(0, 0, 0)$ and its tip at the point (a, b, c) . This vector is obtained by starting at $(0, 0, 0)$, moving parallel to the x axis to $(a, 0, 0)$ and then from here, moving parallel to the y axis to $(a, b, 0)$ and finally parallel to the z axis to (a, b, c) . Observe that the same vector would result if you began at the point (d, e, f) , moved parallel to the x axis to $(d + a, e, f)$, then parallel to the y axis to $(d + a, e + b, f)$, and finally parallel to the z axis to $(d + a, e + b, f + c)$. Here, the vector would have its tail sitting at the point determined by $A = (d, e, f)$ and its point at $B = (d + a, e + b, f + c)$. It is the **same vector** because it will point in the same direction and have the same length. It is like you took an actual arrow, and moved it from one location to another keeping it pointing the same direction.

We conclude this section with a brief discussion regarding notation. In previous sections, we have written vectors as columns, or $n \times 1$ matrices. For convenience in this chapter we may write vectors as the transpose of row vectors, or $1 \times n$ matrices. These are of course equivalent and we may move between both notations. Therefore, recognize that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

Notice that two vectors $\vec{u} = [u_1 \cdots u_n]^T$ and $\vec{v} = [v_1 \cdots v_n]^T$ are equal if and only if all corresponding components are equal. Precisely,

$$\begin{aligned} \vec{u} &= \vec{v} \text{ if and only if} \\ u_j &= v_j \text{ for all } j = 1, \dots, n \end{aligned}$$

Thus $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ and $\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ but $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \neq \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$ because, even though the same numbers are involved, the order of the numbers is different.

For the specific case of \mathbb{R}^3 , there are three special vectors which we often use. They are given by

$$\begin{aligned} \vec{i} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ \vec{j} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \vec{k} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \end{aligned}$$

We can write any vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ as a linear combination of these vectors, written as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. This notation will be used throughout this chapter.

4.2 ALGEBRA IN \mathbb{R}^n

Outcomes

A. Understand vector addition and scalar multiplication, algebraically.

Addition and scalar multiplication are two important algebraic operations done with vectors. Notice that these operations apply to vectors in \mathbb{R}^n , for any value of n . We will explore these operations in more detail in the following sections.

4.2.1. ADDITION OF VECTORS IN \mathbb{R}^n

Addition of vectors in \mathbb{R}^n is defined as follows.

Definition 4.3: Addition of Vectors in \mathbb{R}^n

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ then $\vec{u} + \vec{v} \in \mathbb{R}^n$ and is defined by

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \end{aligned}$$

To add vectors, we simply add corresponding components exactly as we did for matrices. Therefore, in order to add vectors, they must be the same size.

Similarly to matrices, addition of vectors satisfies some important properties. These are outlined in the following theorem.

Theorem 4.4: Properties of Vector Addition

The following properties hold for vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

- The Commutative Law of Addition

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

- The Associative Law of Addition

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

- The Existence of an Additive Identity

$$\vec{u} + \vec{0} = \vec{u} \tag{4.1}$$

- The Existence of an Additive Inverse

$$\vec{u} + (-\vec{u}) = \vec{0}$$

The proof of this theorem follows from the similar theorem given for matrices in Proposition 2.7. Thus the additive identity shown in equation 4.1 is also called the **zero vector**, the $n \times 1$ vector in which all components are equal to 0. Further, $-\vec{u}$ is simply the vector with all components having same value as those of \vec{u} but opposite sign; this is just $(-1)\vec{u}$. This will be made more explicit in the next section when we explore scalar multiplication of vectors. Note that subtraction is defined as $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

4.2.2. SCALAR MULTIPLICATION OF VECTORS IN \mathbb{R}^n

Scalar multiplication of vectors in \mathbb{R}^n is defined as follows. Notice that, just like addition, this definition is the same as the corresponding definition for matrices.

Definition 4.5: Scalar Multiplication of Vectors in \mathbb{R}^n

If $\vec{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u} \in \mathbb{R}^n$ is defined by

$$k\vec{u} = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

Just as with addition, scalar multiplication of vectors satisfies several important properties. These are outlined in the following theorem.

Theorem 4.6: Properties of Scalar Multiplication

The following properties hold for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and k, p scalars.

- The Distributive Law over Vector Addition

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

- The Distributive Law over Scalar Addition

$$(k + p)\vec{u} = k\vec{u} + p\vec{u}$$

- The Associative Law for Scalar Multiplication

$$k(p\vec{u}) = (kp)\vec{u}$$

- Rule for Multiplication by 1

$$1\vec{u} = \vec{u}$$

Proof: Again the verification of these properties follows from the corresponding properties for scalar multiplication of matrices, given in Proposition 2.10.

As a refresher we can show that

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

Note that:

$$\begin{aligned} k(\vec{u} + \vec{v}) &= k[u_1 + v_1 \cdots u_n + v_n]^T \\ &= [k(u_1 + v_1) \cdots k(u_n + v_n)]^T \\ &= [ku_1 + kv_1 \cdots ku_n + kv_n]^T \\ &= [ku_1 \cdots ku_n]^T + [kv_1 \cdots kv_n]^T \\ &= k\vec{u} + k\vec{v} \end{aligned}$$

■

4.2.3. EXERCISES

1. Find $-3 \begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5 \begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$.

2. Find $-7 \begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6 \begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$.

4.3 GEOMETRIC MEANING OF VECTOR ADDITION

Outcomes

A. Understand vector addition, geometrically.

Recall that an element of \mathbb{R}^n is an ordered list of numbers. For the specific case of $n = 2, 3$ this can be used to determine a point in two or three dimensional space. This point is specified relative to some coordinate axes.

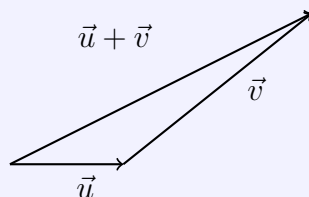
Consider the case $n = 3$. Recall that taking a vector and moving it around without changing its length or direction does not change the vector. This is important in the geometric representation of vector addition.

Suppose we have two vectors, \vec{u} and \vec{v} in \mathbb{R}^3 . Each of these can be drawn geometrically by placing the tail of each vector at 0 and its point at (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. Suppose we slide the vector \vec{v} so that its tail sits at the point of \vec{u} . We know that this does not change the vector \vec{v} . Now, draw a new vector from the tail of \vec{u} to the point of \vec{v} . This vector is $\vec{u} + \vec{v}$.

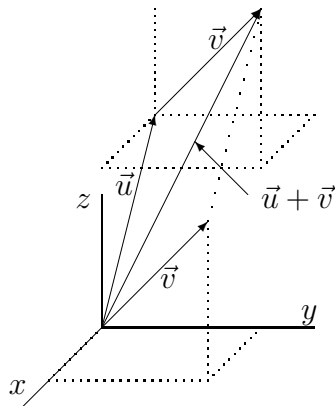
The geometric significance of vector addition in \mathbb{R}^n for any n is given in the following definition.

Definition 4.7: Geometry of Vector Addition

Let \vec{u} and \vec{v} be two vectors. Slide \vec{v} so that the tail of \vec{v} is on the point of \vec{u} . Then draw the arrow which goes from the tail of \vec{u} to the point of \vec{v} . This arrow represents the vector $\vec{u} + \vec{v}$.



This definition is illustrated in the following picture in which $\vec{u} + \vec{v}$ is shown for the special case $n = 3$.

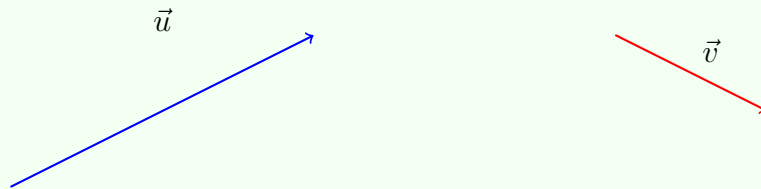


Notice the parallelogram created by \vec{u} and \vec{v} in the above diagram. Then $\vec{u} + \vec{v}$ is the directed diagonal of the parallelogram determined by the two vectors \vec{u} and \vec{v} .

When you have a vector \vec{v} , its additive inverse $-\vec{v}$ will be the vector which has the same magnitude as \vec{v} but the opposite direction. When one writes $\vec{u} - \vec{v}$, the meaning is $\vec{u} + (-\vec{v})$ as with real numbers. The following example illustrates these definitions and conventions.

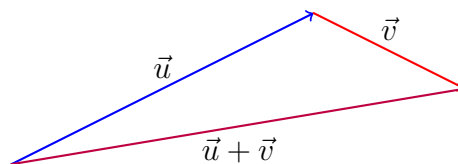
Example 4.8: Graphing Vector Addition

Consider the following picture of vectors \vec{u} and \vec{v} .

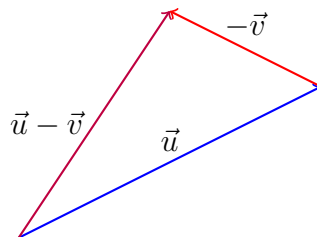


Sketch a picture of $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$.

Solution. We will first sketch $\vec{u} + \vec{v}$. Begin by drawing \vec{u} and then at the point of \vec{u} , place the tail of \vec{v} as shown. Then $\vec{u} + \vec{v}$ is the vector which results from drawing a vector from the tail of \vec{u} to the tip of \vec{v} .



Next consider $\vec{u} - \vec{v}$. This means $\vec{u} + (-\vec{v})$. From the above geometric description of vector addition, $-\vec{v}$ is the vector which has the same length but which points in the opposite direction to \vec{v} . Here is a picture.



□

4.4 LENGTH OF A VECTOR

Outcomes

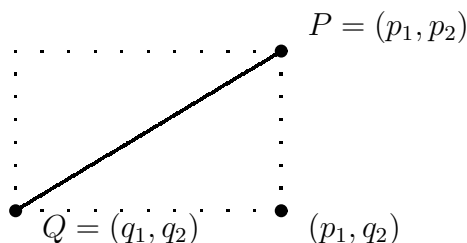
- A. Find the length of a vector and the distance between two points in \mathbb{R}^n .
- B. Find the corresponding unit vector to a vector in \mathbb{R}^n .

In this section, we explore what is meant by the length of a vector in \mathbb{R}^n . We develop this concept by first looking at the distance between two points in \mathbb{R}^n .

First, we will consider the concept of distance for \mathbb{R} , that is, for points in \mathbb{R}^1 . Here, the distance between two points P and Q is given by the absolute value of their difference. We denote the distance between P and Q by $d(P, Q)$ which is defined as

$$d(P, Q) = \sqrt{(P - Q)^2} \quad (4.2)$$

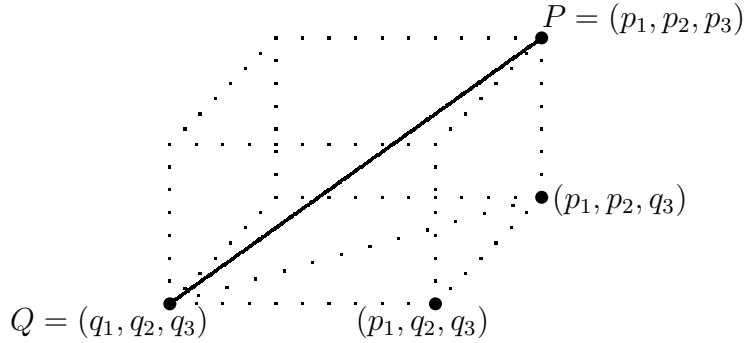
Consider now the case for $n = 2$, demonstrated by the following picture.



There are two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in the plane. The distance between these points is shown in the picture as a solid line. Notice that this line is the hypotenuse of a right triangle which is half of the rectangle shown in dotted lines. We want to find the length of this hypotenuse which will give the distance between the two points. Note the lengths of the sides of this triangle are $|p_1 - q_1|$ and $|p_2 - q_2|$, the absolute value of the difference in these values. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$(|p_1 - q_1|^2 + |p_2 - q_2|^2)^{1/2} = ((p_1 - q_1)^2 + (p_2 - q_2)^2)^{1/2} \quad (4.3)$$

Now suppose $n = 3$ and let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be two points in \mathbb{R}^3 . Consider the following picture in which the solid line joins the two points and a dotted line joins the points (q_1, q_2, q_3) and (p_1, p_2, q_3) .



Here, we need to use Pythagorean Theorem twice in order to find the length of the solid line. First, by the Pythagorean Theorem, the length of the dotted line joining (q_1, q_2, q_3) and (p_1, p_2, q_3) equals

$$\left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2}$$

while the length of the line joining (p_1, p_2, q_3) to (p_1, p_2, p_3) is just $|p_3 - q_3|$. Therefore, by the Pythagorean Theorem again, the length of the line joining the points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ equals

$$\begin{aligned} & \left(\left(\left((p_1 - q_1)^2 + (p_2 - q_2)^2 \right)^{1/2} \right)^2 + (p_3 - q_3)^2 \right)^{1/2} \\ &= \left((p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 \right)^{1/2} \end{aligned} \quad (4.4)$$

This discussion motivates the following definition for the distance between points in \mathbb{R}^n .

Definition 4.9: Distance Between Points

Let $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$ be two points in \mathbb{R}^n . Then the distance between these points is defined as

$$\text{distance between } P \text{ and } Q = d(P, Q) = \left(\sum_{k=1}^n |p_k - q_k|^2 \right)^{1/2}$$

This is called the **distance formula**. We may also write $|P - Q|$ as the distance between P and Q .

From the above discussion, you can see that Definition 4.9 holds for the special cases $n = 1, 2, 3$, as in Equations 4.2, 4.3, 4.4. In the following example, we use Definition 4.9 to find the distance between two points in \mathbb{R}^4 .

Example 4.10: Distance Between Points

Find the distance between the points P and Q in \mathbb{R}^4 , where P and Q are given by

$$P = (1, 2, -4, 6)$$

and

$$Q = (2, 3, -1, 0)$$

Solution. We will use the formula given in Definition 4.9 to find the distance between P and Q . Use the distance formula and write

$$d(P, Q) = ((1 - 2)^2 + (2 - 3)^2 + (-4 - (-1))^2 + (6 - 0)^2)^{\frac{1}{2}} = 47$$

$$\text{Therefore, } d(P, Q) = \sqrt{47}.$$

□

There are certain properties of the distance between points which are important in our study. These are outlined in the following theorem.

Theorem 4.11: Properties of Distance

Let P and Q be points in \mathbb{R}^n , and let the distance between them, $d(P, Q)$, be given as in Definition 4.9. Then, the following properties hold .

- $d(P, Q) = d(Q, P)$
- $d(P, Q) \geq 0$, and equals 0 exactly when $P = Q$.

There are many applications of the concept of distance. For instance, given two points, we can ask what collection of points are all the same distance between the given points. This is explored in the following example.

Example 4.12: The Plane Between Two Points

Describe the points in \mathbb{R}^3 which are at the same distance between $(1, 2, 3)$ and $(0, 1, 2)$.

Solution. Let $P = (p_1, p_2, p_3)$ be such a point. Therefore, P is the same distance from $(1, 2, 3)$ and $(0, 1, 2)$. Then by Definition 4.9,

$$\sqrt{(p_1 - 1)^2 + (p_2 - 2)^2 + (p_3 - 3)^2} = \sqrt{(p_1 - 0)^2 + (p_2 - 1)^2 + (p_3 - 2)^2}$$

Squaring both sides we obtain

$$(p_1 - 1)^2 + (p_2 - 2)^2 + (p_3 - 3)^2 = p_1^2 + (p_2 - 1)^2 + (p_3 - 2)^2$$

and so

$$p_1^2 - 2p_1 + 14 + p_2^2 - 4p_2 + p_3^2 - 6p_3 = p_1^2 + p_2^2 - 2p_2 + 5 + p_3^2 - 4p_3$$

Simplifying, this becomes

$$-2p_1 + 14 - 4p_2 - 6p_3 = -2p_2 + 5 - 4p_3$$

which can be written as

$$2p_1 + 2p_2 + 2p_3 = -9 \quad (4.5)$$

Therefore, the points $P = (p_1, p_2, p_3)$ which are the same distance from each of the given points form a plane whose equation is given by 4.5. \square

We can now use our understanding of the distance between two points to define what is meant by the length of a vector. Consider the following definition.

Definition 4.13: Length of a Vector

Let $\vec{u} = [u_1 \cdots u_n]^T$ be a vector in \mathbb{R}^n . Then, the length of \vec{u} , written $\|\vec{u}\|$ is given by

$$\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

This definition corresponds to Definition 4.9, if you consider the vector \vec{u} to have its tail at the point $0 = (0, \cdots, 0)$ and its tip at the point $U = (u_1, \cdots, u_n)$. Then the length of \vec{u} is equal to the distance between 0 and U , $d(0, U)$. In general, $d(P, Q) = \|\vec{PQ}\|$.

Consider Example 4.10. By Definition 4.13, we could also find the distance between P and Q as the length of the vector connecting them. Hence, if we were to draw a vector \vec{PQ} with its tail at P and its point at Q , this vector would have length equal to $\sqrt{47}$.

We conclude this section with a new definition for the special case of vectors of length 1.

Definition 4.14: Unit Vector

Let \vec{u} be a vector in \mathbb{R}^n . Then, we call \vec{u} a **unit vector** if it has length 1, that is if

$$\|\vec{u}\| = 1$$

Let \vec{v} be a vector in \mathbb{R}^n . Then, the vector \vec{u} which has the same direction as \vec{v} but length equal to 1 is the corresponding unit vector of \vec{v} . This vector is given by

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$$

We often use the term **normalize** to refer to this process. When we **normalize** a vector, we find the corresponding unit vector of length 1. Consider the following example.

Example 4.15: Finding a Unit Vector

Let \vec{v} be given by

$$\vec{v} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T$$

Find the unit vector \vec{u} which has the same direction as \vec{v} .

Solution. We will use Definition 4.14 to solve this. Therefore, we need to find the length of \vec{v} which, by Definition 4.13 is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Using the corresponding values we find that

$$\begin{aligned}\|\vec{v}\| &= \sqrt{1^2 + (-3)^2 + 4^2} \\ &= \sqrt{1 + 9 + 16} \\ &= \sqrt{26}\end{aligned}$$

In order to find \vec{u} , we divide \vec{v} by $\sqrt{26}$. The result is

$$\begin{aligned}\vec{u} &= \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{1}{\sqrt{26}} \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \end{bmatrix}^T\end{aligned}$$

You can verify using the Definition 4.13 that $\|\vec{u}\| = 1$.

□

4.5 GEOMETRIC MEANING OF SCALAR MULTIPLICATION

Outcomes

A. Understand scalar multiplication, geometrically.

Recall that the point $P = (p_1, p_2, p_3)$ determines a vector \vec{p} from 0 to P . The length of \vec{p} , denoted $\|\vec{p}\|$, is equal to $\sqrt{p_1^2 + p_2^2 + p_3^2}$ by Definition 4.9.

Now suppose we have a vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and we multiply \vec{u} by a scalar k . By Definition 4.5, $k\vec{u} = \begin{bmatrix} ku_1 & ku_2 & ku_3 \end{bmatrix}^T$. Then, by using Definition 4.9, the length of this vector is given by

$$\sqrt{((ku_1)^2 + (ku_2)^2 + (ku_3)^2)} = |k| \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Thus the following holds.

$$\|k\vec{u}\| = |k| \|\vec{u}\|$$

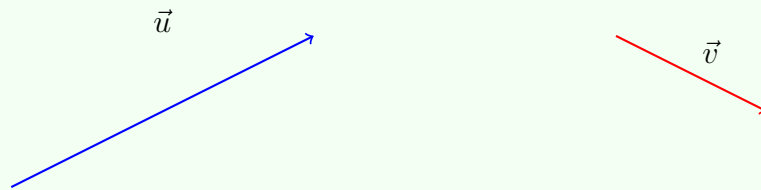
In other words, multiplication by a scalar magnifies or shrinks the length of the vector by a factor of $|k|$. If $|k| > 1$, the length of the resulting vector will be magnified. If $|k| < 1$, the length of the resulting vector will shrink. Remember that by the definition of the absolute value, $|k| > 0$.

What about the direction? Draw a picture of \vec{u} and $k\vec{u}$ where k is negative. Notice that this causes the resulting vector to point in the opposite direction while if $k > 0$ it preserves the direction the vector points. Therefore the direction can either reverse, if $k < 0$, or remain preserved, if $k > 0$.

Consider the following example.

Example 4.16: Graphing Scalar Multiplication

Consider the vectors \vec{u} and \vec{v} drawn below.



Draw $-\vec{u}$, $2\vec{v}$, and $-\frac{1}{2}\vec{v}$.

Solution.

In order to find $-\vec{u}$, we preserve the length of \vec{u} and simply reverse the direction. For $2\vec{v}$, we double the length of \vec{v} , while preserving the direction. Finally $-\frac{1}{2}\vec{v}$ is found by taking half the length of \vec{v} and reversing the direction. These vectors are shown in the following diagram.



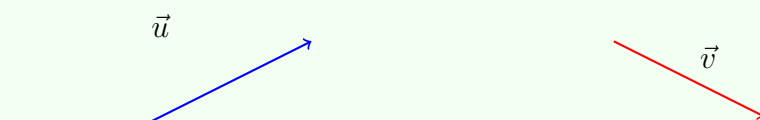
□

Now that we have studied both vector addition and scalar multiplication, we can combine the two actions. Recall Definition 1.32 of linear combinations of column matrices. We can apply this definition to vectors in \mathbb{R}^n . A linear combination of vectors in \mathbb{R}^n is a sum of vectors multiplied by scalars.

In the following example, we examine the geometric meaning of this concept.

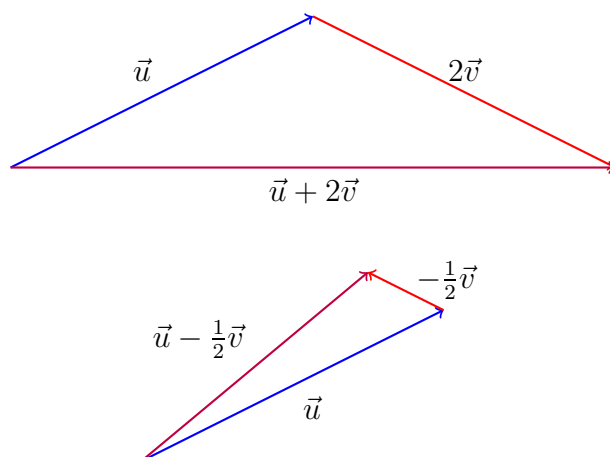
Example 4.17: Graphing a Linear Combination of Vectors

Consider the following picture of the vectors \vec{u} and \vec{v}



Sketch a picture of $\vec{u} + 2\vec{v}$, $\vec{u} - \frac{1}{2}\vec{v}$.

Solution. The two vectors are shown below.



□

4.6 PARAMETRIC LINES

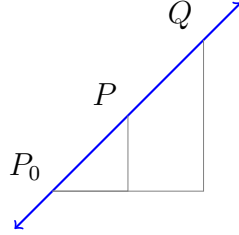
Outcomes

A. Find the vector and parametric equations of a line.

We can use the concept of vectors and points to find equations for arbitrary lines in \mathbb{R}^n , although in this section the focus will be on lines in \mathbb{R}^3 .

To begin, consider the case $n = 1$ so we have $\mathbb{R}^1 = \mathbb{R}$. There is only one line here which is the familiar number line, that is \mathbb{R} itself. Therefore it is not necessary to explore the case of $n = 1$ further.

Now consider the case where $n = 2$, in other words \mathbb{R}^2 . Let P and P_0 be two different points in \mathbb{R}^2 which are contained in a line L . Let \vec{p} and \vec{p}_0 be the position vectors for the points P and P_0 respectively. Suppose that Q is an arbitrary point on L . Consider the following diagram.



Our goal is to be able to define Q in terms of P and P_0 . Consider the vector $\overrightarrow{P_0P} = \vec{p} - \vec{p}_0$ which has its tail at P_0 and point at P . If we add $\vec{p} - \vec{p}_0$ to the position vector \vec{p}_0 for P_0 , the sum would be a vector with its point at P . In other words,

$$\vec{p} = \vec{p}_0 + (\vec{p} - \vec{p}_0)$$

Now suppose we were to add $t(\vec{p} - \vec{p}_0)$ to \vec{p} where t is some scalar. You can see that by doing so, we could find a vector with its point at Q . In other words, we can find t such that

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

This equation determines the line L in \mathbb{R}^2 . In fact, it determines a line L in \mathbb{R}^n . Consider the following definition.

Definition 4.18: Vector Equation of a Line

Suppose a line L in \mathbb{R}^n contains the two different points P and P_0 . Let \vec{p} and \vec{p}_0 be the position vectors of these two points, respectively. Then, L is the collection of points Q which have the position vector \vec{q} given by

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

where $t \in \mathbb{R}$.

Let $\vec{d} = \vec{p} - \vec{p}_0$. Then \vec{d} is the **direction vector for** L and the **vector equation for** L is given by

$$\vec{p} = \vec{p}_0 + t\vec{d}, t \in \mathbb{R}$$

Note that this definition agrees with the usual notion of a line in two dimensions and so this is consistent with earlier concepts. Consider now points in \mathbb{R}^3 . If a point $P \in \mathbb{R}^3$ is given by $P = (x, y, z)$, $P_0 \in \mathbb{R}^3$ by $P_0 = (x_0, y_0, z_0)$, then we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This is the vector equation of L written in **component form**.

The following theorem claims that such an equation is in fact a line.

Proposition 4.19: Algebraic Description of a Straight Line

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ with $\vec{b} \neq \vec{0}$. Then $\vec{x} = \vec{a} + t\vec{b}$, $t \in \mathbb{R}$, is a line.

Proof. Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$. Define $\vec{x}_1 = \vec{a}$ and let $\vec{x}_2 - \vec{x}_1 = \vec{b}$. Since $\vec{b} \neq \vec{0}$, it follows that $\vec{x}_2 \neq \vec{x}_1$. Then $\vec{a} + t\vec{b} = \vec{x}_1 + t(\vec{x}_2 - \vec{x}_1)$. It follows that $\vec{x} = \vec{a} + t\vec{b}$ is a line containing the two different points X_1 and X_2 whose position vectors are given by \vec{x}_1 and \vec{x}_2 respectively. \square

We can use the above discussion to find the equation of a line when given two distinct points. Consider the following example.

Example 4.20: A Line From Two Points

Find a vector equation for the line through the points $P_0 = (1, 2, 0)$ and $P = (2, -4, 6)$.

Solution. We will use the definition of a line given above in Definition 4.18 to write this line in the form

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

Let $\vec{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then, we can find \vec{p} and \vec{p}_0 by taking the position vectors of points P and P_0 respectively. Then,

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}, \quad t \in \mathbb{R}$$

Here, the direction vector $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}$ is obtained by $\vec{p} - \vec{p}_0 = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as indicated above in Definition 4.18. \square

Notice that in the above example we said that we found “a” vector equation for the line, not “the” equation. The reason for this terminology is that there are infinitely many different vector equations for the same line. To see this, replace t with another parameter, say $3s$. Then you obtain a different vector equation for the same line because the same set of points is obtained.

In Example 4.20, the vector given by $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}$ is the direction vector defined in Definition 4.18. If we know the direction vector of a line, as well as a point on the line, we can find the vector equation.

Consider the following example.

Example 4.21: A Line From a Point and a Direction Vector

Find a vector equation for the line which contains the point $P_0 = (1, 2, 0)$ and has direction vector $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solution. We will use Definition 4.18 to write this line in the form $\vec{p} = \vec{p}_0 + t\vec{d}$, $t \in \mathbb{R}$. We are given the direction vector \vec{d} . In order to find \vec{p}_0 , we can use the position vector of the point P_0 . This is given by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Letting $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the equation for the line is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R} \quad (4.6)$$

□

We sometimes elect to write a line such as the one given in 4.6 in the form

$$\left. \begin{array}{l} x = 1 + t \\ y = 2 + 2t \\ z = t \end{array} \right\} \quad \text{where } t \in \mathbb{R} \quad (4.7)$$

This set of equations give the same information as 4.6, and is called the **parametric equation of the line**.

Consider the following definition.

Definition 4.22: Parametric Equation of a Line

Let L be a line in \mathbb{R}^3 which has direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and goes through the point $P_0 = (x_0, y_0, z_0)$. Then, letting t be a parameter, we can write L as

$$\left. \begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array} \right\} \quad \text{where } t \in \mathbb{R}$$

This is called a **parametric equation** of the line L .

You can verify that the form discussed following Example 4.21 in equation 4.7 is of the form given in Definition 4.22.

There is one other form for a line which is useful, which is the **symmetric form**. Consider the line given by 4.7. You can solve for the parameter t to write

$$\begin{aligned}t &= x - 1 \\t &= \frac{y-2}{2} \\t &= z\end{aligned}$$

Therefore,

$$x - 1 = \frac{y - 2}{2} = z$$

This is the **symmetric form** of the line.

In the following example, we look at how to take the equation of a line from symmetric form to parametric form.

Example 4.23: Change Symmetric Form to Parametric Form

Suppose the **symmetric form of a line** is

$$\frac{x - 2}{3} = \frac{y - 1}{2} = z + 3$$

Write the line in parametric form as well as vector form.

Solution. We want to write this line in the form given by Definition 4.22. This is of the form

$$\left. \begin{aligned}x &= x_0 + ta \\y &= y_0 + tb \\z &= z_0 + tc\end{aligned} \right\} \text{ where } t \in \mathbb{R}$$

Let $t = \frac{x-2}{3}$, $t = \frac{y-1}{2}$ and $t = z + 3$, as given in the symmetric form of the line. Then solving for x, y, z , yields

$$\left. \begin{aligned}x &= 2 + 3t \\y &= 1 + 2t \\z &= -3 + t\end{aligned} \right\} \text{ with } t \in \mathbb{R}$$

This is the parametric equation for this line.

Now, we want to write this line in the form given by Definition 4.18. This is the form

$$\vec{p} = \vec{p}_0 + t\vec{d}$$

where $t \in \mathbb{R}$. This equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

□

4.6.1. EXERCISES

1. Find the vector equation for the line through $(-7, 6, 0)$ and $(-1, 1, 4)$. Then, find the parametric equations for this line.
2. Find parametric equations for the line through the point $(7, 7, 1)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.
3. Parametric equations of the line are

$$\begin{aligned}x &= t + 2 \\y &= 6 - 3t \\z &= -t - 6\end{aligned}$$

Find a direction vector for the line and a point on the line.

4. Find the vector equation for the line through the two points $(-5, 5, 1)$, $(2, 2, 4)$. Then, find the parametric equations.
5. The equation of a line in two dimensions is written as $y = x - 5$. Find parametric equations for this line.
6. Find parametric equations for the line through $(6, 5, -2)$ and $(5, 1, 2)$.
7. Find the vector equation and parametric equations for the line through the point $(-7, 10, -6)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.
8. Parametric equations of the line are

$$\begin{aligned}x &= 2t + 2 \\y &= 5 - 4t \\z &= -t - 3\end{aligned}$$

Find a direction vector for the line and a point on the line, and write the vector equation of the line.

9. Find the vector equation and parametric equations for the line through the two points $(4, 10, 0)$, $(1, -5, -6)$.
10. Find the point on the line segment from $P = (-4, 7, 5)$ to $Q = (2, -2, -3)$ which is $\frac{1}{7}$ of the way from P to Q .
11. Suppose a triangle in \mathbb{R}^n has vertices at P_1, P_2 , and P_3 . Consider the lines which are drawn from a vertex to the mid point of the opposite side. Show these three lines intersect in a point and find the coordinates of this point.

4.7 THE DOT PRODUCT

Outcomes

A. Compute the dot product of vectors, and use this to compute vector projections.

4.7.1. THE DOT PRODUCT

There are two ways of multiplying vectors which are of great importance in applications. The first of these is called the **dot product**. When we take the dot product of vectors, the result is a scalar. For this reason, the dot product is also called the **scalar product** and sometimes the **inner product**. The definition is as follows.

Definition 4.24: Dot Product

Let \vec{u}, \vec{v} be two vectors in \mathbb{R}^n . Then we define the **dot product** $\vec{u} \bullet \vec{v}$ as

$$\vec{u} \bullet \vec{v} = \sum_{k=1}^n u_k v_k$$

The dot product $\vec{u} \bullet \vec{v}$ is sometimes denoted as (\vec{u}, \vec{v}) where a comma replaces \bullet . It can also be written as $\langle \vec{u}, \vec{v} \rangle$. If we write the vectors as column or row matrices, it is equal to the matrix product $\vec{v} \vec{u}^T$.

Consider the following example.

Example 4.25: Compute a Dot Product

Find $\vec{u} \bullet \vec{v}$ for

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution. By Definition 4.24, we must compute

$$\vec{u} \bullet \vec{v} = \sum_{k=1}^4 u_k v_k$$

This is given by

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\ &= 0 + 2 + 0 + -3 \\ &= -1\end{aligned}$$

□

With this definition, there are several important properties satisfied by the dot product.

Proposition 4.26: Properties of the Dot Product

Let k and p denote scalars and $\vec{u}, \vec{v}, \vec{w}$ denote vectors. Then the dot product $\vec{u} \bullet \vec{v}$ satisfies the following properties.

- $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
- $\vec{u} \bullet \vec{u} \geq 0$ and equals zero if and only if $\vec{u} = \vec{0}$
- $(k\vec{u} + p\vec{v}) \bullet \vec{w} = k(\vec{u} \bullet \vec{w}) + p(\vec{v} \bullet \vec{w})$
- $\vec{u} \bullet (k\vec{v} + p\vec{w}) = k(\vec{u} \bullet \vec{v}) + p(\vec{u} \bullet \vec{w})$
- $\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$

The proof of this proposition is left as an exercise.

This proposition tells us that we can also use the dot product to find the length of a vector.

Example 4.27: Length of a Vector

Find the length of

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \end{bmatrix}$$

That is, find $\|\vec{u}\|$.

Solution. By Proposition 4.26, $\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$. Therefore, $\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}}$. First, compute $\vec{u} \bullet \vec{u}$. This is given by

$$\begin{aligned}\vec{u} \bullet \vec{u} &= (2)(2) + (1)(1) + (4)(4) + (2)(2) \\ &= 4 + 1 + 16 + 4 \\ &= 25\end{aligned}$$

Then,

$$\begin{aligned}\|\vec{u}\| &= \sqrt{\vec{u} \bullet \vec{u}} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

□

You may wish to compare this to our previous definition of length, given in Definition 4.13.

The **Cauchy Schwarz inequality** is a fundamental inequality satisfied by the dot product. It is given in the following theorem.

Theorem 4.28: Cauchy Schwarz Inequality

The dot product satisfies the inequality

$$|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\| \|\vec{v}\| \quad (4.8)$$

Furthermore equality is obtained if and only if one of \vec{u} or \vec{v} is a scalar multiple of the other.

Proof. First note that if $\vec{v} = \vec{0}$ both sides of 4.8 equal zero and so the inequality holds in this case. Therefore, it will be assumed in what follows that $\vec{v} \neq \vec{0}$.

Define a function of $t \in \mathbb{R}$ by

$$f(t) = (\vec{u} + t\vec{v}) \bullet (\vec{u} + t\vec{v})$$

Then by Proposition 4.26, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Also from Proposition 4.26

$$\begin{aligned}f(t) &= \vec{u} \bullet (\vec{u} + t\vec{v}) + t\vec{v} \bullet (\vec{u} + t\vec{v}) \\ &= \vec{u} \bullet \vec{u} + t(\vec{u} \bullet \vec{v}) + t\vec{v} \bullet \vec{u} + t^2\vec{v} \bullet \vec{v} \\ &= \|\vec{u}\|^2 + 2t(\vec{u} \bullet \vec{v}) + \|\vec{v}\|^2 t^2\end{aligned}$$

Now this means the graph of $y = f(t)$ is a parabola which opens up and either its vertex touches the t axis or else the entire graph is above the t axis. In the first case, there exists some t where $f(t) = 0$ and this requires $\vec{u} + t\vec{v} = \vec{0}$ so one vector is a multiple of the other. Then clearly equality holds in 4.8. In the case where \vec{v} is not a multiple of \vec{u} , it follows $f(t) > 0$ for all t which says $f(t)$ has no real zeros and so from the quadratic formula,

$$(2(\vec{u} \bullet \vec{v}))^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 < 0$$

which is equivalent to $|\vec{u} \bullet \vec{v}| < \|\vec{u}\| \|\vec{v}\|$. □

Notice that this proof was based only on the properties of the dot product listed in Proposition 4.26. This means that whenever an operation satisfies these properties, the

Cauchy Schwarz inequality holds. There are many other instances of these properties besides vectors in \mathbb{R}^n .

The Cauchy Schwarz inequality provides another proof of the **triangle inequality** for distances in \mathbb{R}^n .

Theorem 4.29: Triangle Inequality

For $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad (4.9)$$

and equality holds if and only if one of the vectors is a non-negative scalar multiple of the other.

Also

$$|\|\vec{u}\| - \|\vec{v}\|| \leq \|\vec{u} - \vec{v}\| \quad (4.10)$$

Proof. By properties of the dot product and the Cauchy Schwarz inequality,

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v}) \\ &= (\vec{u} \bullet \vec{u}) + (\vec{u} \bullet \vec{v}) + (\vec{v} \bullet \vec{u}) + (\vec{v} \bullet \vec{v}) \\ &= \|\vec{u}\|^2 + 2(\vec{u} \bullet \vec{v}) + \|\vec{v}\|^2 \\ &\leq \|\vec{u}\|^2 + 2|\vec{u} \bullet \vec{v}| + \|\vec{v}\|^2 \\ &\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

Hence,

$$\|\vec{u} + \vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

Taking square roots of both sides you obtain 4.9.

It remains to consider when equality occurs. Suppose $\vec{u} = \vec{0}$. Then, $\vec{u} = 0\vec{v}$ and the claim about when equality occurs is verified. The same argument holds if $\vec{v} = \vec{0}$. Therefore, it can be assumed both vectors are nonzero. To get equality in 4.9 above, Theorem 4.28 implies one of the vectors must be a multiple of the other. Say $\vec{v} = k\vec{u}$. If $k < 0$ then equality cannot occur in 4.9 because in this case

$$\vec{u} \bullet \vec{v} = k\|\vec{u}\|^2 < 0 < |k| \|\vec{u}\|^2 = |\vec{u} \bullet \vec{v}|$$

Therefore, $k \geq 0$.

To get the other form of the triangle inequality write

$$\vec{u} = \vec{u} - \vec{v} + \vec{v}$$

so

$$\begin{aligned} \|\vec{u}\| &= \|\vec{u} - \vec{v} + \vec{v}\| \\ &\leq \|\vec{u} - \vec{v}\| + \|\vec{v}\| \end{aligned}$$

Therefore,

$$\|\vec{u}\| - \|\vec{v}\| \leq \|\vec{u} - \vec{v}\| \quad (4.11)$$

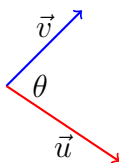
Similarly,

$$\|\vec{v}\| - \|\vec{u}\| \leq \|\vec{v} - \vec{u}\| = \|\vec{u} - \vec{v}\| \quad (4.12)$$

It follows from 4.11 and 4.12 that 4.10 holds. This is because $||\vec{u}\| - \|\vec{v}\||$ equals the left side of either 4.11 or 4.12 and either way, $||\vec{u}\| - \|\vec{v}\|| \leq \|\vec{u} - \vec{v}\|$. \square

4.7.2. THE GEOMETRIC SIGNIFICANCE OF THE DOT PRODUCT

Given two vectors, \vec{u} and \vec{v} , the **included angle** is the angle between these two vectors which is less than or equal to 180 degrees. The dot product can be used to determine the included angle between two vectors. Consider the following picture where θ gives the included angle.



Proposition 4.30: The Dot Product and the Included Angle

Let \vec{u} and \vec{v} be two vectors in \mathbb{R}^n , and let θ be the included angle. Then the following equation holds.

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

In words, the dot product of two vectors equals the product of the magnitude (or length) of the two vectors multiplied by the cosine of the included angle. Note this gives a geometric description of the dot product which does not depend explicitly on the coordinates of the vectors.

Consider the following example.

Example 4.31: Find the Angle Between Two Vectors

Find the angle between the vectors given by

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Solution. By Proposition 4.30,

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

Hence,

$$\cos \theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

First, we can compute $\vec{u} \bullet \vec{v}$. By Definition 4.24, this equals

$$\vec{u} \bullet \vec{v} = (2)(3) + (1)(4) + (-1)(1) = 9$$

Then,

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(2)(2) + (1)(1) + (1)(1)} = \sqrt{6} \\ \|\vec{v}\| &= \sqrt{(3)(3) + (4)(4) + (1)(1)} = \sqrt{26}\end{aligned}$$

Therefore, the cosine of the included angle equals

$$\cos \theta = \frac{9}{\sqrt{26}\sqrt{6}} = 0.7205766\dots$$

With the cosine known, the angle can be determined by computing the inverse cosine of that angle, giving approximately $\theta = 0.76616$ radians. \square

Another application of the geometric description of the dot product is in finding the angle between two lines. Typically one would assume that the lines intersect. In some situations, however, it may make sense to ask this question when the lines do not intersect, such as the angle between two object trajectories. In any case we understand it to mean the smallest angle between (any of) their direction vectors. The only subtlety here is that if \vec{u} is a direction vector for a line, then so is any multiple $k\vec{u}$, and thus we will find complementary angles among all angles between direction vectors for two lines, and we simply take the smaller of the two.

Example 4.32: Find the Angle Between Two Lines

Find the angle between the two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

and

$$L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Solution. You can verify that these lines do not intersect, but as discussed above this does not matter and we simply find the smallest angle between any directions vectors for these lines.

To do so we first find the angle between the direction vectors given above:

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

In order to find the angle, we solve the following equation for θ

$$\vec{u} \bullet \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos \theta$$

to obtain $\cos \theta = -\frac{1}{2}$ and since we choose included angles between 0 and π we obtain $\theta = \frac{2\pi}{3}$.

Now the angles between any two direction vectors for these lines will either be $\frac{2\pi}{3}$ or its complement $\phi = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. We choose the smaller angle, and therefore conclude that the angle between the two lines is $\frac{\pi}{3}$. \square

We can also use Proposition 4.30 to compute the dot product of two vectors.

Example 4.33: Using Geometric Description to Find a Dot Product

Let \vec{u}, \vec{v} be vectors with $\|\vec{u}\| = 3$ and $\|\vec{v}\| = 4$. Suppose the angle between \vec{u} and \vec{v} is $\pi/3$. Find $\vec{u} \bullet \vec{v}$.

Solution. From the geometric description of the dot product in Proposition 4.30

$$\vec{u} \bullet \vec{v} = (3)(4) \cos(\pi/3) = 3 \times 4 \times 1/2 = 6$$

\square

Two nonzero vectors are said to be **perpendicular**, sometimes also called **orthogonal**, if the included angle is $\pi/2$ radians (90°).

Consider the following proposition.

Proposition 4.34: Perpendicular Vectors

Let \vec{u} and \vec{v} be nonzero vectors in \mathbb{R}^n . Then, \vec{u} and \vec{v} are said to be **perpendicular** exactly when

$$\vec{u} \bullet \vec{v} = 0$$

Proof. This follows directly from Proposition 4.30. First if the dot product of two nonzero vectors is equal to 0, this tells us that $\cos \theta = 0$ (this is where we need nonzero vectors). Thus $\theta = \pi/2$ and the vectors are perpendicular.

If on the other hand \vec{v} is perpendicular to \vec{u} , then the included angle is $\pi/2$ radians. Hence $\cos \theta = 0$ and $\vec{u} \bullet \vec{v} = 0$. \square

Consider the following example.

Example 4.35: Determine if Two Vectors are Perpendicular

Determine whether the two vectors,

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

are perpendicular.

Solution. In order to determine if these two vectors are perpendicular, we compute the dot product. This is given by

$$\vec{u} \bullet \vec{v} = (2)(1) + (1)(3) + (-1)(5) = 0$$

Therefore, by Proposition 4.34 these two vectors are perpendicular. \square

4.7.3. PROJECTIONS

In some applications, we wish to write a vector as a sum of two related vectors. Through the concept of projections, we can find these two vectors. First, we explore an important theorem. The result of this theorem will provide our definition of a vector projection.

Theorem 4.36: Vector Projections

Let \vec{v} and \vec{u} be nonzero vectors. Then there exist unique vectors $\vec{v}_{||}$ and \vec{v}_{\perp} such that

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp} \tag{4.13}$$

where $\vec{v}_{||}$ is a scalar multiple of \vec{u} , and \vec{v}_{\perp} is perpendicular to \vec{u} .

Proof. Suppose 4.13 holds and $\vec{v}_{||} = k\vec{u}$. Taking the dot product of both sides of 4.13 with \vec{u} and using $\vec{v}_{\perp} \bullet \vec{u} = 0$, this yields

$$\begin{aligned} \vec{v} \bullet \vec{u} &= (\vec{v}_{||} + \vec{v}_{\perp}) \bullet \vec{u} \\ &= k\vec{u} \bullet \vec{u} + \vec{v}_{\perp} \bullet \vec{u} \\ &= k\|\vec{u}\|^2 \end{aligned}$$

which requires $k = \vec{v} \bullet \vec{u} / \|\vec{u}\|^2$. Thus there can be no more than one vector $\vec{v}_{||}$. It follows \vec{v}_{\perp} must equal $\vec{v} - \vec{v}_{||}$. This verifies there can be no more than one choice for both $\vec{v}_{||}$ and \vec{v}_{\perp} and proves their uniqueness.

Now let

$$\vec{v}_{||} = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2} \vec{u}$$

and let

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{||} = \vec{v} - \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2} \vec{u}$$

Then $\vec{v}_{||} = k\vec{u}$ where $k = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}$. It only remains to verify $\vec{v}_{\perp} \bullet \vec{u} = 0$. But

$$\begin{aligned} \vec{v}_{\perp} \bullet \vec{u} &= \vec{v} \bullet \vec{u} - \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2} \vec{u} \bullet \vec{u} \\ &= \vec{v} \bullet \vec{u} - \vec{v} \bullet \vec{u} \\ &= 0 \end{aligned}$$

\square

The vector $\vec{v}_{||}$ in Theorem 4.36 is called the **projection** of \vec{v} onto \vec{u} and is denoted by

$$\vec{v}_{||} = \text{proj}_{\vec{u}}(\vec{v})$$

We now make a formal definition of the vector projection.

Definition 4.37: Vector Projection

Let \vec{u} and \vec{v} be vectors. Then, the **projection of \vec{v} onto \vec{u}** is given by

$$\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u} = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2} \vec{u}$$

Consider the following example of a projection.

Example 4.38: Find the Projection of One Vector Onto Another

Find $\text{proj}_{\vec{u}}(\vec{v})$ if

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Solution. We can use the formula provided in Definition 4.37 to find $\text{proj}_{\vec{u}}(\vec{v})$. First, compute $\vec{v} \bullet \vec{u}$. This is given by

$$\begin{aligned} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(1) + (3)(-2) + (-4)(1) \\ &= 2 - 6 - 4 \\ &= -8 \end{aligned}$$

Similarly, $\vec{u} \bullet \vec{u}$ is given by

$$\begin{aligned} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(2) + (3)(3) + (-4)(-4) \\ &= 4 + 9 + 16 \\ &= 29 \end{aligned}$$

Therefore, the projection is equal to

$$\begin{aligned} \text{proj}_{\vec{u}}(\vec{v}) &= -\frac{8}{29} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{16}{29} \\ -\frac{24}{29} \\ \frac{32}{29} \end{bmatrix} \end{aligned}$$

□

We will conclude this section with an important application of projections. Suppose a line L and a point P are given such that P is not contained in L . Through the use of projections, we can determine the shortest distance from P to L .

Example 4.39: Shortest Distance from a Point to a Line

Let $P = (1, 3, 5)$ be a point in \mathbb{R}^3 , and let L be the line which goes through point $P_0 = (0, 4, -2)$ with direction vector $\vec{d} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the shortest distance from P to the line L , and find the point Q on L that is closest to P .

Solution. In order to determine the shortest distance from P to L , we will first find the vector $\overrightarrow{P_0P}$ and then find the projection of this vector onto L . The vector $\overrightarrow{P_0P}$ is given by

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

Then, if Q is the point on L closest to P , it follows that

$$\begin{aligned} \overrightarrow{P_0Q} &= \text{proj}_{\vec{d}} \overrightarrow{P_0P} \\ &= \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{\|\vec{d}\|^2} \right) \vec{d} \\ &= \frac{15}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Now, the distance from P to L is given by

$$\|\overrightarrow{QP}\| = \|\overrightarrow{P_0P} - \overrightarrow{P_0Q}\| = \sqrt{26}$$

The point Q is found by adding the vector $\overrightarrow{P_0Q}$ to the position vector $\overrightarrow{0P_0}$ for P_0 as follows

$$\begin{aligned} \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} &= \frac{2}{3} \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{3} \\ \frac{20}{3} \\ \frac{4}{3} \end{bmatrix} \end{aligned}$$

Therefore, $Q = (\frac{10}{3}, \frac{20}{3}, \frac{4}{3})$.

□

4.7.4. EXERCISES

1. Find $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

2. Use the formula given in Proposition 4.30 to verify the Cauchy Schwarz inequality and to show that equality occurs if and only if one of the vectors is a scalar multiple of the other.

3. For \vec{u}, \vec{v} vectors in \mathbb{R}^3 , define the product, $\vec{u} * \vec{v} = u_1v_1 + 2u_2v_2 + 3u_3v_3$. Show the axioms for a dot product all hold for this product. Prove

$$\|\vec{u} * \vec{v}\| \leq (\vec{u} * \vec{u})^{1/2} (\vec{v} * \vec{v})^{1/2}$$

4. Let \vec{a}, \vec{b} be vectors. Show that $(\vec{a} \bullet \vec{b}) = \frac{1}{4} (\|\vec{a} + \vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2)$.

5. Using the axioms of the dot product, prove the parallelogram identity:

$$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$$

6. Let A be a real $m \times n$ matrix and let $\vec{u} \in \mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^m$. Show $A\vec{u} \bullet \vec{v} = \vec{u} \bullet A^T\vec{v}$.
Hint: Use the definition of matrix multiplication to do this.

7. Use the result of Problem 6 to verify directly that $(AB)^T = B^T A^T$ without making any reference to subscripts.

8. Find the angle between the vectors

$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

9. Find the angle between the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

10. Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

11. Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.
12. Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.
13. Let $P = (1, 2, 3)$ be a point in \mathbb{R}^3 . Let L be the line through the point $P_0 = (1, 4, 5)$ with direction vector $\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find the shortest distance from P to L , and find the point Q on L that is closest to P .
14. Let $P = (0, 2, 1)$ be a point in \mathbb{R}^3 . Let L be the line through the point $P_0 = (1, 1, 1)$ with direction vector $\vec{d} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Find the shortest distance from P to L , and find the point Q on L that is closest to P .
15. Does it make sense to speak of $\text{proj}_{\vec{0}}(\vec{w})$?
16. Prove the Cauchy Schwarz inequality in \mathbb{R}^n as follows. For \vec{u}, \vec{v} vectors, consider

$$(\vec{w} - \text{proj}_{\vec{v}}\vec{w}) \bullet (\vec{w} - \text{proj}_{\vec{v}}\vec{w}) \geq 0$$

Simplify using the axioms of the dot product and then put in the formula for the projection. Notice that this expression equals 0 and you get equality in the Cauchy Schwarz inequality if and only if $\vec{w} = \text{proj}_{\vec{v}}\vec{w}$. What is the geometric meaning of $\vec{w} = \text{proj}_{\vec{v}}\vec{w}$?

17. Let $\vec{v}, \vec{w}, \vec{u}$ be vectors. Show that $(\vec{w} + \vec{u})_{\perp} = \vec{w}_{\perp} + \vec{u}_{\perp}$ where $\vec{w}_{\perp} = \vec{w} - \text{proj}_{\vec{v}}(\vec{w})$.
18. Show that

$$(\vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u}) = (\vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u}) = (\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) \bullet \vec{u} = 0$$

and conclude every vector in \mathbb{R}^n can be written as the sum of two vectors, one which is perpendicular and one which is parallel to the given vector.

4.8 PLANES IN \mathbb{R}^n

Outcomes

A. Find the vector and scalar equations of a plane.

Much like the above discussion with lines, vectors can be used to determine planes in \mathbb{R}^n . Given a vector \vec{n} in \mathbb{R}^n and a point P_0 , it is possible to find a **unique** plane which contains P_0 and is perpendicular to the given vector.

Definition 4.40: Normal Vector

Let \vec{n} be a nonzero vector in \mathbb{R}^n . Then \vec{n} is called a **normal vector** to a plane if and only if

$$\vec{n} \bullet \vec{v} = 0$$

for every vector \vec{v} in the plane.

In other words, we say that \vec{n} is orthogonal (perpendicular) to every vector in the plane.

Consider now a plane with normal vector given by \vec{n} , and containing a point P_0 . Notice that this plane is unique. If P is an arbitrary point on this plane, then by definition the normal vector is orthogonal to the vector between P_0 and P . Letting $\vec{0P}$ and $\vec{0P_0}$ be the position vectors of points P and P_0 respectively, it follows that

$$\vec{n} \bullet (\vec{0P} - \vec{0P_0}) = 0$$

or

$$\vec{n} \bullet \vec{P_0P} = 0$$

The first of these equations gives the **vector equation** of the plane.

Definition 4.41: Vector Equation of a Plane

Let \vec{n} be the normal vector for a plane which contains a point P_0 . If P is an arbitrary point on this plane, then the **vector equation** of the plane is given by

$$\vec{n} \bullet (\vec{0P} - \vec{0P_0}) = 0$$

Notice that this equation can be used to determine if a point P is contained in a certain plane.

Example 4.42: A Point in a Plane

Let $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be the normal vector for a plane which contains the point $P_0 = (2, 1, 4)$.
Determine if the point $P = (5, 4, 1)$ is contained in this plane.

Solution. By Definition 4.41, P is a point in the plane if it satisfies the equation

$$\vec{n} \bullet (\overrightarrow{OP} - \overrightarrow{OP_0}) = 0$$

Given the above \vec{n} , P_0 , and P , this equation becomes

$$\begin{aligned} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right) \\ &= 3 + 6 - 9 = 0 \end{aligned}$$

Therefore $P = (5, 4, 1)$ is contained in the plane. □

Suppose $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$.

Then

$$\begin{aligned} \vec{n} \bullet (\overrightarrow{OP} - \overrightarrow{OP_0}) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned}$$

We can also write this equation as

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Notice that since P_0 is given, $ax_0 + by_0 + cz_0$ is a known scalar, which we can call d . This equation becomes

$$ax + by + cz = d$$

Definition 4.43: Scalar Equation of a Plane

Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the normal vector for a plane which contains the point $P_0 = (x_0, y_0, z_0)$. Then if $P = (x, y, z)$ is an arbitrary point on the plane, the **scalar equation** of the plane is given by

$$ax + by + cz = d$$

where $a, b, c, d \in \mathbb{R}$ and $d = ax_0 + by_0 + cz_0$.

Consider the following equation.

Example 4.44: Finding the Equation of a Plane

Find an equation of the plane containing $P_0 = (3, -2, 5)$ and orthogonal to $\vec{n} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

Solution. The above vector \vec{n} is the normal vector for this plane. Using Definition 4.41, we can determine the vector equation for this plane.

$$\begin{aligned} \vec{n} \bullet (\vec{OP} - \vec{OP}_0) &= 0 \\ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x-3 \\ y+2 \\ z-5 \end{bmatrix} &= 0 \end{aligned}$$

Using Definition 4.43, we can determine the scalar equation of the plane.

$$-2x + 4y + 1z = -2(3) + 4(-2) + 1(5) = -9$$

Hence, the vector equation of the plane is

$$\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x-3 \\ y+2 \\ z-5 \end{bmatrix} = 0$$

and the scalar equation is

$$-2x + 4y + 1z = -9$$

□

Suppose a point P is not contained in a given plane. We are then interested in the shortest distance from that point P to the given plane. Consider the following example.

Example 4.45: Shortest Distance From a Point to a Plane

Find the shortest distance from the point $P = (3, 2, 3)$ to the plane given by $2x + y + 2z = 2$, and find the point Q on the plane that is closest to P .

Solution. Pick an arbitrary point P_0 on the plane. Then, it follows that

$$\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P}$$

and $\|\overrightarrow{QP}\|$ is the shortest distance from P to the plane. Further, the vector $\overrightarrow{OQ} = \overrightarrow{OP} - \overrightarrow{QP}$ gives the necessary point Q .

From the above scalar equation, we have that $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Now, choose $P_0 = (1, 0, 0)$ so that $\vec{n} \bullet \overrightarrow{OP_0} = 2 = d$. Then, $\overrightarrow{P_0P} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.
Next, compute $\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P}$.

$$\begin{aligned} \overrightarrow{QP} &= \text{proj}_{\vec{n}} \overrightarrow{P_0P} \\ &= \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} \\ &= \frac{12}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Then, $\|\overrightarrow{QP}\| = 4$ so the shortest distance from P to the plane is 4.

Next, to find the point Q on the plane which is closest to P we have

$$\begin{aligned} \overrightarrow{OQ} &= \overrightarrow{OP} - \overrightarrow{QP} \\ &= \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $Q = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$. □

4.9 THE CROSS PRODUCT

Outcomes

A. Compute the cross product and box product of vectors in \mathbb{R}^3 .

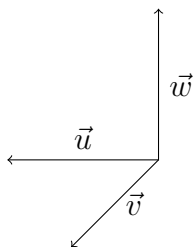
Recall that the dot product is one of two important products for vectors. The second type of product for vectors is called the cross product. It is important to note that the cross product is only defined in \mathbb{R}^3 . First we discuss the geometric meaning and then a description in terms of coordinates is given, both of which are important. The geometric description is essential in order to understand the applications to physics and geometry while the coordinate description is necessary to compute the cross product.

Consider the following definition.

Definition 4.46: Right Hand System of Vectors

Three vectors, $\vec{u}, \vec{v}, \vec{w}$ form a right hand system if when you extend the fingers of your right hand along the direction of vector \vec{u} and close them in the direction of \vec{v} , the thumb points roughly in the direction of \vec{w} .

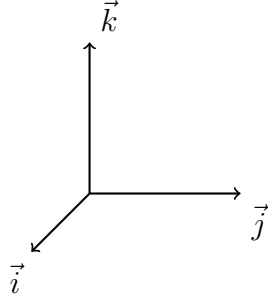
For an example of a right handed system of vectors, see the following picture.



In this picture the vector \vec{w} points upwards from the plane determined by the other two vectors. Point the fingers of your right hand along \vec{u} , and close them in the direction of \vec{v} . Notice that if you extend the thumb on your right hand, it points in the direction of \vec{w} .

You should consider how a right hand system would differ from a left hand system. Try using your left hand and you will see that the vector \vec{w} would need to point in the opposite direction.

Notice that the special vectors, $\vec{i}, \vec{j}, \vec{k}$ will always form a right handed system. If you extend the fingers of your right hand along \vec{i} and close them in the direction \vec{j} , the thumb points in the direction of \vec{k} .



The following is the geometric description of the cross product. Recall that the dot product of two vectors results in a scalar. In contrast, the cross product results in a vector, as the product gives a direction as well as magnitude.

Definition 4.47: Geometric Definition of Cross Product

Let \vec{u} and \vec{v} be two vectors in \mathbb{R}^3 . Then the **cross product**, written $\vec{u} \times \vec{v}$, is defined by the following two rules.

1. Its length is $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$, where θ is the included angle between \vec{u} and \vec{v} .
2. It is perpendicular to both \vec{u} and \vec{v} , that is $(\vec{u} \times \vec{v}) \bullet \vec{u} = 0$, $(\vec{u} \times \vec{v}) \bullet \vec{v} = 0$, and $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ form a right hand system.

The cross product of the special vectors $\vec{i}, \vec{j}, \vec{k}$ is as follows.

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} & \vec{j} \times \vec{i} &= -\vec{k} \\ \vec{k} \times \vec{i} &= \vec{j} & \vec{i} \times \vec{k} &= -\vec{j} \\ \vec{j} \times \vec{k} &= \vec{i} & \vec{k} \times \vec{j} &= -\vec{i} \end{aligned}$$

With this information, the following gives the coordinate description of the cross product.

Recall that the vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ can be written in terms of $\vec{i}, \vec{j}, \vec{k}$ as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$.

Proposition 4.48: Coordinate Description of Cross Product

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be two vectors. Then

$$\begin{aligned} \vec{u} \times \vec{v} &= (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + \\ &\quad + (u_1v_2 - u_2v_1)\vec{k} \end{aligned} \tag{4.14}$$

Writing $\vec{u} \times \vec{v}$ in the usual way, it is given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

We now prove this proposition.

Proof. From the above table and the properties of the cross product listed,

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\
&= u_1v_2\vec{i} \times \vec{j} + u_1v_3\vec{i} \times \vec{k} + u_2v_1\vec{j} \times \vec{i} + u_2v_3\vec{j} \times \vec{k} + u_3v_1\vec{k} \times \vec{i} + u_3v_2\vec{k} \times \vec{j} \\
&= u_1v_2\vec{k} - u_1v_3\vec{j} - u_2v_1\vec{k} + u_2v_3\vec{i} + u_3v_1\vec{j} - u_3v_2\vec{i} \\
&= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}
\end{aligned} \tag{4.15}$$

□

There is another version of 4.14 which may be easier to remember. We can express the cross product as the determinant of a matrix, as follows.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{4.16}$$

Expanding the determinant along the top row yields

$$\begin{aligned}
&\vec{i}(-1)^{1+1} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \vec{j}(-1)^{2+1} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}(-1)^{3+1} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}
\end{aligned}$$

Expanding these determinants leads to

$$(u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

which is the same as 4.15.

The cross product satisfies the following properties.

Proposition 4.49: Properties of the Cross Product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in \mathbb{R}^3 , and k a scalar. Then, the following properties of the cross product hold.

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$, and $\vec{u} \times \vec{u} = \vec{0}$
2. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

Proof. Formula 1. follows immediately from the definition. The vectors $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ have the same magnitude, $|\vec{u}| |\vec{v}| \sin \theta$, and an application of the right hand rule shows they have opposite direction.

Formula 2. is proven as follows. If k is a non-negative scalar, the direction of $(k\vec{u}) \times \vec{v}$ is the same as the direction of $\vec{u} \times \vec{v}$, $k(\vec{u} \times \vec{v})$ and $\vec{u} \times (k\vec{v})$. The magnitude is k times the magnitude of $\vec{u} \times \vec{v}$ which is the same as the magnitude of $k(\vec{u} \times \vec{v})$ and $\vec{u} \times (k\vec{v})$. Using this yields equality in 2. In the case where $k < 0$, everything works the same way except the vectors are all pointing in the opposite direction and you must multiply by $|k|$ when comparing their magnitudes.

The distributive laws, 3. and 4., are much harder to establish. For now, it suffices to notice that if we know that 3. is true, 4. follows. Thus, assuming 3., and using 1.,

$$\begin{aligned} (\vec{v} + \vec{w}) \times \vec{u} &= -\vec{u} \times (\vec{v} + \vec{w}) \\ &= -(\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \\ &= \vec{v} \times \vec{u} + \vec{w} \times \vec{u} \end{aligned}$$

□

We will now look at an example of how to compute a cross product.

Example 4.50: Find a Cross Product

Find $\vec{u} \times \vec{v}$ for the following vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution. Note that we can write \vec{u}, \vec{v} in terms of the special vectors $\vec{i}, \vec{j}, \vec{k}$ as

$$\begin{aligned} \vec{u} &= \vec{i} - \vec{j} + 2\vec{k} \\ \vec{v} &= 3\vec{i} - 2\vec{j} + \vec{k} \end{aligned}$$

We will use the equation given by 4.16 to compute the cross product.

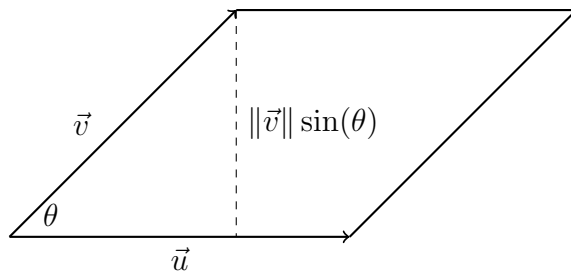
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \vec{k} = 3\vec{i} + 5\vec{j} + \vec{k}$$

We can write this result in the usual way, as

$$\vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

□

An important geometrical application of the cross product is as follows. The size of the cross product, $\|\vec{u} \times \vec{v}\|$, is the area of the parallelogram determined by \vec{u} and \vec{v} , as shown in the following picture.



We examine this concept in the following example.

Example 4.51: Area of a Parallelogram

Find the area of the parallelogram determined by the vectors \vec{u} and \vec{v} given by

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution. Notice that these vectors are the same as the ones given in Example 4.50. Recall from the geometric description of the cross product, that the area of the parallelogram is simply the magnitude of $\vec{u} \times \vec{v}$. From Example 4.50, $\vec{u} \times \vec{v} = 3\vec{i} + 5\vec{j} + \vec{k}$. We can also write this as

$$\vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Thus the area of the parallelogram is

$$\|\vec{u} \times \vec{v}\| = \sqrt{(3)(3) + (5)(5) + (1)(1)} = \sqrt{9 + 25 + 1} = \sqrt{35}$$

□

We can also use this concept to find the area of a triangle. Consider the following example.

Example 4.52: Area of Triangle

Find the area of the triangle determined by the points $(1, 2, 3)$, $(0, 2, 5)$, $(5, 1, 2)$

Solution. This triangle is obtained by connecting the three points with lines. Picking $(1, 2, 3)$ as a starting point, there are two displacement vectors, $\begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 4 & -1 & -1 \end{bmatrix}^T$. Notice that if we add either of these vectors to the position vector of the starting point, the result is the position vectors of the other two points. Now, the area of the triangle is half the area of the parallelogram determined by $\begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 4 & -1 & -1 \end{bmatrix}^T$. The required cross product is given by

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix}$$

Taking the size of this vector gives the area of the parallelogram, given by

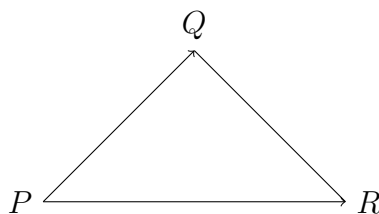
$$\sqrt{(2)(2) + (7)(7) + (1)(1)} = \sqrt{4 + 49 + 1} = \sqrt{54}$$

Hence the area of the triangle is $\frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6}$. \square

In general, if you have three points in \mathbb{R}^3 , P, Q, R , the area of the triangle is given by

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

Recall that \vec{PQ} is the vector running from point P to point Q .



In the next section, we explore another application of the cross product.

4.9.1. THE BOX PRODUCT

Recall that we can use the cross product to find the area of a parallelogram. It follows that we can use the cross product together with the dot product to find the volume of a parallelepiped.

We begin with a definition.

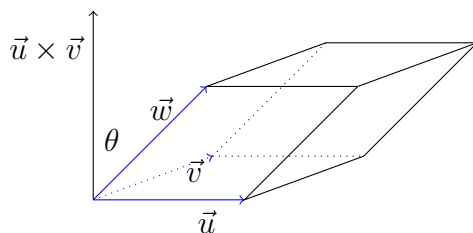
Definition 4.53: Parallelepiped

A parallelepiped determined by the three vectors, \vec{u}, \vec{v} , and \vec{w} consists of

$$\{r\vec{u} + s\vec{v} + t\vec{w} : r, s, t \in [0, 1]\}$$

That is, if you pick three numbers, r, s , and t each in $[0, 1]$ and form $r\vec{u} + s\vec{v} + t\vec{w}$ then the collection of all such points makes up the parallelepiped determined by these three vectors.

The following is an example of a parallelepiped.



Notice that the base of the parallelepiped is the parallelogram determined by the vectors \vec{u} and \vec{v} . Therefore, its area is equal to $\|\vec{u} \times \vec{v}\|$. The height of the parallelepiped is $\|\vec{w}\| \cos \theta$ where θ is the angle shown in the picture between \vec{w} and $\vec{u} \times \vec{v}$. The volume of this parallelepiped is the area of the base times the height which is just

$$\|\vec{u} \times \vec{v}\| \|\vec{w}\| \cos \theta = \vec{u} \times \vec{v} \bullet \vec{w}$$

This expression is known as the box product and is sometimes written as $[\vec{u}, \vec{v}, \vec{w}]$. You should consider what happens if you interchange the \vec{v} with the \vec{w} or the \vec{u} with the \vec{w} . You can see geometrically from drawing pictures that this merely introduces a minus sign. In any case the box product of three vectors always equals either the volume of the parallelepiped determined by the three vectors or else -1 times this volume.

Proposition 4.54: The Box Product

Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in \mathbb{R}^n that define a parallelepiped. Then the volume of the parallelepiped is the absolute value of the box product, given by

$$|\vec{u} \times \vec{v} \bullet \vec{w}|$$

Consider an example of this concept.

Example 4.55: Volume of a Parallelepiped

Find the volume of the parallelepiped determined by the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

Solution. According to the above discussion, pick any two of these vectors, take the cross product and then take the dot product of this with the third of these vectors. The result will be either the desired volume or -1 times the desired volume. Therefore by taking the absolute value of the result, we obtain the volume.

We will take the cross product of \vec{u} and \vec{v} . This is given by

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -5 \\ 1 & 3 & -6 \end{vmatrix} = 3\vec{i} + \vec{j} + \vec{k} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Now take the dot product of this vector with \vec{w} which yields

$$\begin{aligned}
 (\vec{u} \times \vec{v}) \bullet \vec{w} &= \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \\
 &= (3\vec{i} + \vec{j} + \vec{k}) \bullet (3\vec{i} + 2\vec{j} + 3\vec{k}) \\
 &= 9 + 2 + 3 \\
 &= 14
 \end{aligned}$$

This shows the volume of this parallelepiped is 14 cubic units. \square

There is a fundamental observation which comes directly from the geometric definitions of the cross product and the dot product.

Proposition 4.56: Order of the Product

Let \vec{u}, \vec{v} , and \vec{w} be vectors. Then $(\vec{u} \times \vec{v}) \bullet \vec{w} = \vec{u} \bullet (\vec{v} \times \vec{w})$.

Proof. This follows from observing that either $(\vec{u} \times \vec{v}) \bullet \vec{w}$ and $\vec{u} \bullet (\vec{v} \times \vec{w})$ both give the volume of the parallelepiped or they both give -1 times the volume. \square

Recall that we can express the cross product as the determinant of a particular matrix. It turns out that the same can be done for the box product. Suppose you have three vectors, $\vec{u} = [a \ b \ c]^T$, $\vec{v} = [d \ e \ f]^T$, and $\vec{w} = [g \ h \ i]^T$. Then the box product $\vec{u} \bullet \vec{v} \times \vec{w}$ is given by the following.

$$\begin{aligned}
 \vec{u} \bullet \vec{v} \times \vec{w} &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ d & e & f \\ g & h & i \end{vmatrix} \\
 &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
 &= \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
 \end{aligned}$$

To take the box product, you can simply take the determinant of the matrix which results by letting the rows be the components of the given vectors in the order in which they occur in the box product.

This follows directly from the definition of the cross product given above and the way we expand determinants. Thus the volume of a parallelepiped determined by the vectors $\vec{u}, \vec{v}, \vec{w}$ is just the absolute value of the above determinant.

4.9.2. EXERCISES

1. Show that if $\vec{a} \times \vec{u} = \vec{0}$ for any unit vector \vec{u} , then $\vec{a} = \vec{0}$.

2. Find the area of the triangle determined by the three points, $(1, 2, 3)$, $(4, 2, 0)$ and $(-3, 2, 1)$.
3. Find the area of the triangle determined by the three points, $(1, 0, 3)$, $(4, 1, 0)$ and $(-3, 1, 1)$.
4. Find the area of the triangle determined by the three points, $(1, 2, 3)$, $(2, 3, 4)$ and $(3, 4, 5)$. Did something interesting happen here? What does it mean geometrically?

5. Find the area of the parallelogram determined by the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

6. Find the area of the parallelogram determined by the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$.

7. Is $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$? What is the meaning of $\vec{u} \times \vec{v} \times \vec{w}$? Explain. **Hint:** Try $(\vec{i} \times \vec{j}) \times \vec{k}$.

8. Verify directly that the coordinate description of the cross product, $\vec{u} \times \vec{v}$ has the property that it is perpendicular to both \vec{u} and \vec{v} . Then show by direct computation that this coordinate description satisfies

$$\begin{aligned} \|\vec{u} \times \vec{v}\|^2 &= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2 \\ &= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2(\theta)) \end{aligned}$$

where θ is the angle included between the two vectors. Explain why $\|\vec{u} \times \vec{v}\|$ has the correct magnitude.

9. Suppose A is a 3×3 skew symmetric matrix such that $A^T = -A$. Show there exists a vector $\vec{\Omega}$ such that for all $\vec{u} \in \mathbb{R}^3$

$$A\vec{u} = \vec{\Omega} \times \vec{u}$$

Hint: Explain why, since A is skew symmetric it is of the form

$$A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

where the ω_i are numbers. Then consider $\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$.

10. Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$,

$$\begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}.$$

11. Suppose \vec{u}, \vec{v} , and \vec{w} are three vectors whose components are all integers. Can you conclude the volume of the parallelepiped determined from these three vectors will always be an integer?
12. What does it mean geometrically if the box product of three vectors gives zero?
13. Using Problem 12, find an equation of a plane containing the two position vectors, \vec{p} and \vec{q} and the point 0. **Hint:** If (x, y, z) is a point on this plane, the volume of the parallelepiped determined by (x, y, z) and the vectors \vec{p}, \vec{q} equals 0.
14. Using the notion of the box product yielding either plus or minus the volume of the parallelepiped determined by the given three vectors, show that

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = \vec{u} \bullet (\vec{v} \times \vec{w})$$

In other words, the dot and the cross can be switched as long as the order of the vectors remains the same. **Hint:** There are two ways to do this, by the coordinate description of the dot and cross product and by geometric reasoning.

15. Simplify $(\vec{u} \times \vec{v}) \bullet (\vec{v} \times \vec{w}) \times (\vec{w} \times \vec{z})$.
16. Simplify $\|\vec{u} \times \vec{v}\|^2 + (\vec{u} \bullet \vec{v})^2 - \|\vec{u}\|^2 \|\vec{v}\|^2$.
17. For $\vec{u}, \vec{v}, \vec{w}$ functions of t , prove the following product rules:

$$\begin{aligned}(\vec{u} \times \vec{v})' &= \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}' \\ (\vec{u} \bullet \vec{v})' &= \vec{u}' \bullet \vec{v} + \vec{u} \bullet \vec{v}'\end{aligned}$$

4.10 SPANNING, LINEAR INDEPENDENCE AND BASIS IN \mathbb{R}^n

Outcomes

- A. Determine the span of a set of vectors, and determine if a vector is contained in a specified span.
- B. Determine if a set of vectors is linearly independent.
- C. Understand the concepts of subspace, basis, and dimension.
- D. Find the row space, column space, and null space of a matrix.

By generating all linear combinations of a set of vectors one can obtain various subsets of \mathbb{R}^n which we call subspaces. For example what set of vectors in \mathbb{R}^3 generate the XY -plane? What is the smallest such set of vectors can you find? The tools of spanning, linear independence and basis are exactly what is needed to answer these and similar questions and are the focus of this section.

4.10.1. SPANNING SET OF VECTORS

We begin this section with a definition.

Definition 4.57: Span of a Set of Vectors

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is known as the span of these vectors and is written as $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$.

Consider the following example.

Example 4.58: Span of Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$.

Solution. You can see that any linear combination of the vectors \vec{u} and \vec{v} yields a vector $\begin{bmatrix} x & y & 0 \end{bmatrix}^T$ in the XY -plane.

Moreover every vector in the XY -plane is in fact such a linear combination of the vectors \vec{u} and \vec{v} . That's because

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (-2x + 3y) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x - y) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

An arbitrary vector in the XY -plane can be written as a linear combination of \vec{u} and \vec{v} . Thus $\text{span}\{\vec{u}, \vec{v}\}$ is precisely the XY -plane. \square

You can convince yourself that no single vector can span the XY -plane. In fact, take a moment to consider what is meant by the span of a single vector.

However you can make the set larger if you wish. For example consider the larger set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ where $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$. Since the first two vectors already span the entire XY -plane, the span is once again precisely the XY -plane and nothing has been gained. Of course if you add a new vector such as $\vec{w} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ then it does span a different space. What is the span of $\vec{u}, \vec{v}, \vec{w}$ in this case?

The distinction between the sets $\{\vec{u}, \vec{v}\}$ and $\{\vec{u}, \vec{v}, \vec{w}\}$ will be made using the concept of linear independence.

Consider the vectors \vec{u}, \vec{v} , and \vec{w} discussed above. In the next example, we will show how to formally demonstrate that \vec{w} is in the span of \vec{u} and \vec{v} .

Example 4.59: Vector in a Span

Let $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$. Show that $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ is in $\text{span}\{\vec{u}, \vec{v}\}$.

Solution. For a vector to be in $\text{span}\{\vec{u}, \vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, we must be able to find scalars a, b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

We proceed as follows.

$$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

This is equivalent to the following system of equations

$$\begin{aligned} a + 3b &= 4 \\ a + 2b &= 5 \end{aligned}$$

We solve this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array} \right]$$

The solution is $a = 7, b = -1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that \vec{w} is in $\text{span}\{\vec{u}, \vec{v}\}$. □

4.10.2. LINEARLY INDEPENDENT SET OF VECTORS

Together with the notion of a spanning set of vectors, linear independence is a very important property of a set of vectors.

Definition 4.60: Linearly Independent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is said to be **linearly independent** if no vector in that set is in the span of the other vectors of that set.

Here is an example.

Example 4.61: Linearly Independent Vectors

Consider the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ in \mathbb{R}^3 . Verify whether the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution. We already verified in Example 4.89 that $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$. Therefore the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is **not** linearly independent. In this case we say it is linearly dependent. \square

In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors. In the previous example you can see that the vector \vec{w} does not help to span any new vector not already in the span of the other two vectors. However you can verify that the set $\{\vec{u}, \vec{v}\}$ is linearly independent, since both are required to span the XY -plane.

Consider the following important theorem.

Theorem 4.62: Linear Independence as a Linear Combination

The collection of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is linearly independent if and only if whenever

$$\sum_{i=1}^n a_i \vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$.

In other words, $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is linearly independent exactly when the system of linear equations $AX = 0$ has only the trivial solution, where A is the $n \times k$ matrix having these vectors as columns.

Proof. Suppose first $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent. Then by Definition 4.90 none of the vectors is a linear combination of the others. Now suppose for the sake of a contradiction that

$$\sum_{i=1}^n a_i \vec{u}_i = \vec{0}$$

and not all the $a_i = 0$. Then pick the a_i which is not zero and divide this equation by it. Solve for \vec{u}_i in terms of the other \vec{u}_j , contradicting the fact that none of the \vec{u}_i equals a linear combination of the others. Therefore if the set of vectors is linearly independent and a linear combination of these vectors equals the zero vector, then all the coefficients must equal zero.

Now suppose a linear combination of the vectors equals the zero vector such that all coefficients equal zero. We want to show that the vectors are linearly independent. If \vec{u}_i is a linear combination of the other vectors in the list, then you could obtain an equation of the form

$$\vec{u}_i = \sum_{j \neq i} a_j \vec{u}_j$$

and so we could write

$$\vec{0} = \sum_{j \neq i} a_j \vec{u}_j + (-1) \vec{u}_i$$

contradicting the condition that all coefficients equal 0.

Finally observe that the expression $\sum_{i=1}^n a_i \vec{u}_i = \vec{0}$ can be written as the system of linear equations $AX = 0$ where A is the $n \times k$ matrix having these vectors as columns. \square

Sometimes we refer to this last condition about sums as follows: The set of vectors, $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals zero. A nontrivial linear combination is one in which not all the scalars equal zero. Similarly, a trivial linear combination is one in which all scalars equal zero.

We can say that a set of vectors is **linearly dependent** if it is not linearly independent, and hence if at least one vector is a linear combination of the others.

Here is a detailed example in \mathbb{R}^4 .

Example 4.63: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution. Form the 4×4 matrix A having these vectors as columns:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Then by Theorem 4.62, the given set of vectors is linearly independent exactly if the system $AX = 0$ has only the trivial solution.

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Not all the columns of the coefficient matrix are pivot columns and so the vectors are not linearly independent. In this case, we say the vectors are linearly dependent.

It follows that there are infinitely many solutions to $AX = 0$, one of which is

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Therefore we can write

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be rearranged as follows

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

This gives the last vector as a linear combination of the first three vectors.

Notice that we could rearrange this equation to write any of the four vectors as a linear combination of the other three. \square

Consider another example.

Example 4.64: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution. In this case the matrix of the corresponding homogeneous system of linear equations is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

and so every column is a pivot column. Therefore, these vectors are linearly independent and there is no way to obtain one of the vectors as a linear combination of the others. \square

The following corollary follows from the fact that if the augmented matrix of a homogeneous system of linear equations has more columns than rows, the system has infinitely many solutions.

Corollary 4.65: Linearly Dependence in \mathbb{R}^n

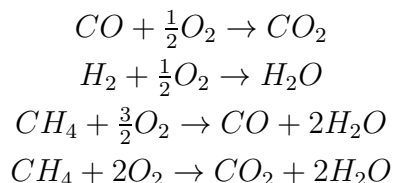
Let $\{\vec{u}_1, \dots, \vec{u}_k\}$ be a set of vectors in \mathbb{R}^n . If $k > n$, then the set is linearly dependent (i.e. NOT linearly independent).

Proof. Form the $n \times k$ matrix A having the vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ as its columns and suppose $k > n$. Then A has rank $r \leq n < k$, so the system $AX = 0$ has a nontrivial solution by Theorem 1.35, and thus not linearly independent by Theorem 4.62. \square

4.10.3. A SHORT APPLICATION TO CHEMISTRY

The following section applies the concepts of spanning and linear independence to the subject of chemistry.

When working with chemical reactions, there are sometimes a large number of reactions and some are in a sense redundant. Suppose you have the following chemical reactions.



There are four chemical reactions here but they are not independent reactions. There is some redundancy. What are the independent reactions? Is there a way to consider a shorter list of reactions? To analyze this situation, we can write the reactions in a matrix as follows

$$\begin{bmatrix} CO & O_2 & CO_2 & H_2 & H_2O & CH_4 \\ 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

Each row contains the coefficients of the respective elements in each reaction. For example, the top row of numbers comes from $CO + \frac{1}{2}O_2 - CO_2 = 0$ which represents the first of the chemical reactions.

We can write these coefficients in the following matrix

$$\begin{bmatrix} 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

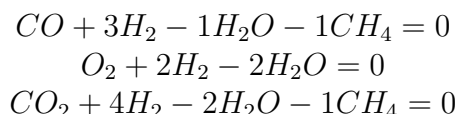
Rather than listing all of the reactions as above, it would be more efficient to only list those which are independent by throwing out that which is redundant. We can use the concepts of the previous section to accomplish this.

First, take the reduced row-echelon form of the above matrix.

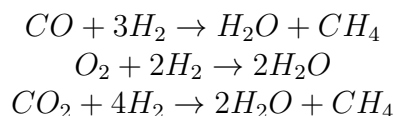
$$\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The top three rows represent “independent” reactions which come from the original four reactions. One can obtain each of the original four rows of the matrix given above by taking a suitable linear combination of rows of this reduced row-echelon form matrix.

With the redundant reaction removed, we can consider the simplified reactions as the following equations



In terms of the original notation, these are the reactions



These three reactions provide an equivalent system to the original four equations. The idea is that, in terms of what happens chemically, you obtain the same information with the shorter list of reactions. Such a simplification is especially useful when dealing with very large lists of reactions which may result from experimental evidence.

4.10.4. SUBSPACES AND BASIS

A subspace is simply a set of vectors with the property that linear combinations of these vectors remain in the set. Geometrically, subspaces are represented by lines and planes which contain the origin. The precise definition is as follows.

Definition 4.66: Subspace

Let V be a nonempty collection of vectors in \mathbb{R}^n . Then V is called a subspace if whenever a and b are scalars and \vec{u} and \vec{v} are vectors in V , the linear combination $a\vec{u} + b\vec{v}$ is also in V .

More generally this means that a subspace contains the span of any finite collection of vectors in that subspace. It turns out that in \mathbb{R}^n , a subspace is exactly the span of finitely many of its vectors.

Theorem 4.67: Subspaces are Spans

Let V be a nonempty collection of vectors in \mathbb{R}^n . Then V is a subspace of \mathbb{R}^n if and only if there exist vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in V such that

$$V = \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$$

Proof. Pick a vector \vec{u}_1 in V . If $V = \text{span}\{\vec{u}_1\}$, then you have found your list of vectors and are done. If $V \neq \text{span}\{\vec{u}_1\}$, then there exists \vec{u}_2 a vector of V which is not in $\text{span}\{\vec{u}_1\}$. Consider $\text{span}\{\vec{u}_1, \vec{u}_2\}$. If $V = \text{span}\{\vec{u}_1, \vec{u}_2\}$, we are done. Otherwise, pick \vec{u}_3 not in $\text{span}\{\vec{u}_1, \vec{u}_2\}$. Continue this way. Note that since V is a subspace, these spans are each contained in V . The process must stop with \vec{u}_k for some $k \leq n$ by Corollary 4.65.

Now suppose $V = \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$, we must show this is a subspace. So let $\sum_{i=1}^k c_i \vec{u}_i$ and $\sum_{i=1}^k d_i \vec{u}_i$ be two vectors in V , and let a and b be two scalars. Then

$$a \sum_{i=1}^k c_i \vec{u}_i + b \sum_{i=1}^k d_i \vec{u}_i = \sum_{i=1}^k (ac_i + bd_i) \vec{u}_i$$

which is one of the vectors in $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$ and is therefore contained in V . This shows that $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$ has the properties of a subspace. \square

Since the vectors \vec{u}_i we constructed in the proof above are not in the span of the previous vectors (by definition), they must be linearly independent and thus we obtain the following corollary.

Corollary 4.68: Subspaces are Spans of Independent Vectors

If V is a subspace of \mathbb{R}^n , then there exist linearly independent vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in V such that $V = \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$.

In summary, subspaces of \mathbb{R}^n consist of spans of finite, linearly independent collections of vectors of \mathbb{R}^n .

Note that it was just shown in Corollary 4.68 that every subspace of \mathbb{R}^n is equal to the span of a linearly independent collection of vectors of \mathbb{R}^n . Such a collection of vectors is called a basis.

Definition 4.69: Basis of a Subspace

Let V be a subspace of \mathbb{R}^n . Then $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\} = V$
2. $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent

Note the plural of basis is **bases**.

The main theorem about bases is not only they exist, but that they must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Theorem.

Theorem 4.70: Exchange Theorem

Suppose $\{\vec{u}_1, \dots, \vec{u}_r\}$ is linearly independent and each \vec{u}_k is contained in $\text{span}\{\vec{v}_1, \dots, \vec{v}_s\}$. Then $s \geq r$. In words, spanning sets have at least as many vectors as linearly independent sets.

Proof. Since $\{\vec{v}_1, \dots, \vec{v}_s\}$ is a spanning set, there exist scalars a_{ij} such that

$$\vec{u}_j = \sum_{i=1}^s a_{ij} \vec{v}_i$$

Suppose for a contradiction that $s < r$. Then the matrix $A = [a_{ij}]$ has fewer rows, s than columns, r . By Theorem 1.35 there exists a non trivial solution \vec{d} such that $\vec{d} \neq \vec{0}$ but $A\vec{d} = \vec{0}$. In other words,

$$\sum_{j=1}^r a_{ij} d_j = 0, \quad i = 1, 2, \dots, s$$

Therefore,

$$\begin{aligned} \sum_{j=1}^r d_j \vec{u}_j &= \sum_{j=1}^r d_j \sum_{i=1}^s a_{ij} \vec{v}_i \\ &= \sum_{i=1}^s \left(\sum_{j=1}^r a_{ij} d_j \right) \vec{v}_i = \sum_{i=1}^s 0 \vec{v}_i = \vec{0} \end{aligned}$$

which contradicts the assumption that $\{\vec{u}_1, \dots, \vec{u}_r\}$ is linearly independent, because not all the $d_j = 0$. Thus this contradiction indicates that $s \geq r$. \square

We are now ready to show that any two bases are of the same size.

Theorem 4.71: Bases of \mathbb{R}^n are of the Same Size

Let V be a subspace of \mathbb{R}^n and suppose $\{\vec{u}_1, \dots, \vec{u}_k\}$ and $\{\vec{v}_1, \dots, \vec{v}_m\}$ are two bases for V . Then $k = m$.

Proof. This follows right away from Theorem 9.27. Indeed observe that $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a spanning set while $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent so $k \geq m$. Also $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a spanning set while $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent so $m \geq k$. \square

The following definition can now be stated.

Definition 4.72: Dimension of a Subspace

Let V be a subspace of \mathbb{R}^n . Then the **dimension** of V , written $\dim(V)$ is defined to be the number of vectors in a basis.

The next result follows.

Corollary 4.73: Dimension of \mathbb{R}^n

The dimension of \mathbb{R}^n is n .

Proof. You only need to exhibit a basis for \mathbb{R}^n which has n vectors. Such a basis is $\{\vec{e}_1, \dots, \vec{e}_n\}$. \square

We conclude this section by stating further properties of a set of vectors in \mathbb{R}^n .

Corollary 4.74: Linearly Independent and Spanning Sets in \mathbb{R}^n

The following properties hold in \mathbb{R}^n :

- Suppose $\{\vec{u}_1, \dots, \vec{u}_n\}$ is linearly independent and each \vec{u}_i is a vector in \mathbb{R}^n . Then $\{\vec{u}_1, \dots, \vec{u}_n\}$ is a basis for \mathbb{R}^n .
- Suppose $\{\vec{u}_1, \dots, \vec{u}_m\}$ spans \mathbb{R}^n . Then $m \geq n$.
- If $\{\vec{u}_1, \dots, \vec{u}_n\}$ spans \mathbb{R}^n , then $\{\vec{u}_1, \dots, \vec{u}_n\}$ is linearly independent.

Proof. Assume first that $\{\vec{u}_1, \dots, \vec{u}_n\}$ is linearly independent, and we need to show that this set spans \mathbb{R}^n . To do so, let \vec{v} be a vector of \mathbb{R}^n , and we need to write \vec{v} as a linear combination of \vec{u}_i 's. Consider the matrix A having the vectors \vec{u}_i as columns:

$$A = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

By linear independence of the \vec{u}_i 's, the reduced row-echelon form of A is the identity matrix. Therefore the system $A\vec{x} = \vec{v}$ has a (unique) solution for all \vec{v} , so \vec{v} is a linear combination of the \vec{u}_i 's.

To establish the second claim, suppose that $m < n$. Then letting $\vec{v}_{i_1}, \dots, \vec{v}_{i_k}$ be the pivot columns of the matrix

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}$$

it follows $k \leq m < n$ and these k pivot columns would be a basis for \mathbb{R}^n having fewer than n vectors, contrary to Corollary 4.73.

Finally consider the third claim. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is not linearly independent, then replace this list with $\{\vec{v}_{i_1}, \dots, \vec{v}_{i_k}\}$ where these are the pivot columns of the matrix

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

Then $\{\vec{v}_{i_1}, \dots, \vec{v}_{i_k}\}$ spans \mathbb{R}^n and is linearly independent so it is a basis having less than n vectors again contrary to Corollary 4.73. \square

4.10.5. ROW SPACE, COLUMN SPACE, AND NULL SPACE OF A MATRIX

We begin this section with a new definition.

Definition 4.75: Row and Column Space

Let A be an $m \times n$ matrix. The **column space** of A , written $\text{col}(A)$, is the span of the columns. The **row space** of A , written $\text{row}(A)$, is the span of the rows.

Using the reduced row-echelon form, we can obtain an efficient description of the row and column space of a matrix. Of course the column space can be obtained by simply saying that it equals the span of all the columns. However, you can often get the column space as the span of fewer columns than this. This is what we mean by an efficient description. The next example illustrates this concept.

Example 4.76: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces efficiently.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Solution. The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the rank is 2. Notice that all columns of this reduced row-echelon form matrix are in

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

For example,

$$\begin{bmatrix} -9 \\ 5 \\ 0 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since the original matrix and its reduced row-echelon form are equivalent, all columns of the original matrix are similarly contained in the span of the first two columns of that matrix. For example, consider the third column of the original matrix. It can be written as a linear combination of the first two columns of the original matrix as follows.

$$\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

The column space of the original matrix equals the span of the first two columns of the **original matrix**. This is the desired efficient description of the column space.

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

What about an efficient description of the row space? When row operations are used, the resulting vectors remain in the row space. Thus the rows in the reduced row-echelon form are in the row space of the original matrix. Furthermore, by reversing the row operations, each row of the original matrix can be obtained as a linear combination of the rows in the reduced row-echelon form. It follows that the span of the nonzero rows in the reduced row-echelon form equals the span of the original rows. For the above matrix, the row space equals

$$\text{row}(A) = \text{span} \{ [1 \ 0 \ -9 \ 9 \ 2], [0 \ 1 \ 5 \ -3 \ 0] \}$$

□

Notice that the column space of A is given as the span of columns of the original matrix, while the row space of A is the span of rows of the reduced row-echelon form of A .

Consider another example.

Example 4.77: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces efficiently.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}$$

Solution. The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the rank is 3. The row space is given by

$$\text{row}(A) = \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & -\frac{5}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & \frac{1}{2} \end{bmatrix} \right\}$$

Notice that the first three columns of the reduced row-echelon form are pivot columns. The column space is the span of the first three columns in the **original matrix**,

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \\ 2 \end{bmatrix} \right\}$$

□

Consider the solution given above for Example 4.77, where the rank of A equals 3. Notice that the row space and the column space each had dimension equal to 3. It turns out that this is not a coincidence. This essential result is referred to as the Rank Theorem and is given now.

Theorem 4.78: Rank Theorem

Let A be an $m \times n$ matrix. Then

$$\dim(\text{col}(A)) = \dim(\text{row}(A)) = \text{rank}(A)$$

The following statements are results of the Rank Theorem.

Corollary 4.79: Results of the Rank Theorem

1. For any matrix A , $\text{rank}(A) = \text{rank}(A^T)$.
2. For any $m \times n$ matrix A , $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
3. For any $n \times n$ matrix A , A is invertible if and only if $\text{rank}(A) = n$.
4. For any matrix A and invertible matrices B, C of appropriate size, $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AC)$.

Consider the following example.

Example 4.80: Rank of the Transpose

Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Find $\text{rank}(A)$ and $\text{rank}(A^T)$.

Solution. To find $\text{rank}(A)$ we first row reduce to find the reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore the rank of A is 2. Now consider A^T given by

$$A^T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Again we row reduce to find the reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You can see that $\text{rank}(A^T) = 2$, the same as $\text{rank}(A)$. □

We now define what is meant by the null space of a general $m \times n$ matrix.

Definition 4.81: Null Space, or Kernel, of A

The null space of a matrix A , also referred to as the kernel of A , is defined as follows.

$$\ker(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$$

It is also referred to quite often using the notation $N(A)$, the N signifying null space. To find $\ker(A)$ one must solve the system of equations $A\vec{x} = \vec{0}$. This is a familiar procedure.

Similarly, we can discuss the image of A , denoted by $\text{im}(A)$. The image of A consists of the vectors of \mathbb{R}^m which “get hit” by A . The formal definition is as follows.

Definition 4.82: Image of A

The image of A , written $\text{im}(A)$ is given by

$$\text{im}(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

Consider A as a mapping from \mathbb{R}^n to \mathbb{R}^m whose action is given by multiplication. The following diagram displays this scenario.

$$\begin{array}{ccc} \ker(A) & & \text{im}(A) \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array}$$

As indicated, $\text{im}(A)$ is a subset of \mathbb{R}^m while $\ker(A)$ is a subset of \mathbb{R}^n .

Finding $\ker(A)$ is not new! There is just some new terminology being used. $\ker(A)$ is simply the solution to the system $A\vec{x} = \vec{0}$.

Consider the following example.

Example 4.83: Null Space, or Kernel of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\ker(A)$.

Solution. In order to find $\ker(A)$, we need to solve the equation $A\vec{x} = \vec{0}$. This is the usual procedure of writing the augmented matrix, finding the reduced row-echelon form and then the solution. The augmented matrix and corresponding reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The third column is not a pivot column, and therefore the solution will contain a parameter. The solution to the system $A\vec{x} = \vec{0}$ is given by

$$\begin{bmatrix} 3t \\ t \\ t \end{bmatrix} : t \in \mathbb{R}$$

which can be written as

$$t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}$$

Therefore, the null space of A is all multiples of this vector, which we can write as

$$\ker(A) = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

□

Here is a larger example, but the method is entirely similar.

Example 4.84: Null Space, or Kernel of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 3 & 0 \\ 3 & 1 & 2 & 3 & 1 \\ 4 & -2 & 2 & 6 & 0 \end{bmatrix}$$

Find the null space of A .

Solution. To find the null space, we need to solve the equation $AX = 0$. The augmented matrix and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 2 & 3 & 1 & 0 \\ 4 & -2 & 2 & 6 & 0 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{5} & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

It follows that the first two columns are pivot columns, and the next three correspond to parameters. Therefore, $\ker(A)$ is given by

$$\begin{bmatrix} \left(-\frac{3}{5}\right)s + \left(-\frac{6}{5}\right)t + \left(\frac{1}{5}\right)r \\ \left(-\frac{1}{5}\right)s + \left(\frac{3}{5}\right)t + \left(-\frac{2}{5}\right)r \\ s \\ t \\ r \end{bmatrix} : s, t, r \in \mathbb{R}.$$

We write this in the form

$$s \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} : s, t, r \in \mathbb{R}.$$

In other words, the null space of this matrix equals the span of the three vectors above. Thus

$$\ker(A) = \text{span} \left\{ \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

□

Notice also that the three vectors above are linearly independent and so the dimension of $\ker(A)$ is 3. The following is true in general. The number of free variables equals the dimension of the null space while the number of basic variables equals the number of pivot columns which is the rank.

Before we proceed to an important theorem, we first define what is meant by the nullity of a matrix.

Definition 4.85: Nullity

The dimension of the null space of a matrix is called the **nullity**, denoted $\text{null}(A)$.

We can now state an important theorem.

Theorem 4.86: Rank and Nullity

Let A be an $m \times n$ matrix. Then $\text{rank}(A) + \text{null}(A) = n$.

Consider the following example, which we first explored above in Example 4.83

Example 4.87: Rank and Nullity

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\text{rank}(A)$ and $\text{null}(A)$.

Solution. In the above Example 4.83 we determined that the reduced row-echelon form of A is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the rank of A is 2. We also determined that kernel of A is given by

$$\ker(A) = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore the nullity of A is 1. It follows from Theorem 4.86 that $\text{rank}(A) + \text{null}(A) = 2 + 1 = 3$, which is the number of columns of A . \square

4.10.6. EXERCISES

1. Suppose $\{\vec{x}_1, \dots, \vec{x}_k\}$ is a set of vectors from \mathbb{R}^n . Show that $\vec{0}$ is in $\text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$.

2. Let $H = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

3. Let H denote $\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

4. Let H denote $\text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -9 \\ 4 \\ 3 \\ -9 \end{bmatrix}, \begin{bmatrix} -33 \\ 15 \\ 12 \\ -36 \end{bmatrix}, \begin{bmatrix} -22 \\ 10 \\ 8 \\ -24 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

5. Let H denote $\text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ -3 \\ -6 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

6. Let H denote $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 15 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 15 \\ 6 \\ 3 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

7. Let H denote $\text{span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 16 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -3 \\ 22 \\ 0 \\ -8 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

8. Let H denote $\text{span} \left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 14 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 38 \\ 8 \\ 6 \\ 24 \end{bmatrix}, \begin{bmatrix} 47 \\ 10 \\ 7 \\ 28 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \\ 3 \\ 12 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

9. Let H denote $\text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 17 \\ 3 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} 52 \\ 9 \\ 7 \\ 35 \end{bmatrix}, \begin{bmatrix} 18 \\ 3 \\ 4 \\ 20 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

10. Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \sin(u_1) = 1 \right\}$. Is M a subspace? Explain.

11. Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : |u_1| \leq 4 \right\}$. Is M a subspace? Explain.

12. Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4 \right\}$. Is M a subspace? Explain.

13. Let \vec{w}, \vec{w}_1 be given vectors in \mathbb{R}^4 and define

$$M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \text{ and } \vec{w}_1 \bullet \vec{u} = 0 \right\}.$$

Is M a subspace? Explain.

14. Let $\vec{w} \in \mathbb{R}^4$ and let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \right\}$. Is M a subspace? Explain.

15. Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 \geq u_1 \right\}$. Is M a subspace? Explain.

16. Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

17. Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 4u + v - 5w \\ 12u + 6v - 6w \\ 4u + 4v + 4w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

18. Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u + 6v + 7w \\ -3u - 9v - 12w \\ 2u + 6v + 6w \\ u + 3v + 3w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of \mathbb{R}^4 ? If so, explain why, give a basis for the subspace and find its dimension.

19. Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u + v \\ 6v - 3u + 3w \\ 3v - 6u + 3w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

20. Consider the vectors of the form

$$\left\{ \begin{bmatrix} 2u + v + 7w \\ u - 2v + w \\ -6v - 6w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

21. Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u + v + 11w \\ 18u + 6v + 66w \\ 28u + 8v + 100w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

22. Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u + v \\ 2w - 4u \\ 2w - 2v - 8u \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

23. Consider the set of vectors S given by

$$\left\{ \begin{bmatrix} u + v + w \\ 2u + 2v + 4w \\ u + v + w \\ 0 \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of \mathbb{R}^4 ? If so, explain why, give a basis for the subspace and find its dimension.

24. Consider the set of vectors S given by

$$\left\{ \begin{bmatrix} v \\ -3u - 3w \\ 8u - 4v + 4w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of \mathbb{R}^3 ? If so, explain why, give a basis for the subspace and find its dimension.

25. If you have 5 vectors in \mathbb{R}^5 and the vectors are linearly independent, can it always be concluded they span \mathbb{R}^5 ? Explain.
26. If you have 6 vectors in \mathbb{R}^5 , is it possible they are linearly independent? Explain.
27. Suppose A is an $m \times n$ matrix and $\{\vec{w}_1, \dots, \vec{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{R}^n) \subseteq \mathbb{R}^m$. Now suppose $A\vec{z}_i = \vec{w}_i$. Show $\{\vec{z}_1, \dots, \vec{z}_k\}$ is also independent.
28. Suppose V, W are subspaces of \mathbb{R}^n . Let $V \cap W$ be all vectors which are in both V and W . Show that $V \cap W$ is a subspace also.
29. Suppose V and W both have dimension equal to 7 and they are subspaces of \mathbb{R}^{10} . What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linear independent set can be extended to form a basis.
30. Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linearly independent set can be extended to form a basis.
31. Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Hint: Consider the subspace, $B(\mathbb{R}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\vec{w}_1, \dots, \vec{w}_k\}$. Now suppose $\{\vec{u}_1, \dots, \vec{u}_r\}$ is a basis for $\ker(B)$. Let $\{\vec{z}_1, \dots, \vec{z}_k\}$ be such that $B\vec{z}_i = \vec{w}_i$ and argue that

$$\ker(AB) \subseteq \text{span}\{\vec{u}_1, \dots, \vec{u}_r, \vec{z}_1, \dots, \vec{z}_k\}.$$

32. Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace of \mathbb{R}^n .
33. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 0 & 3 \\ 3 & 9 & 1 & -7 & 0 & 8 \\ 1 & 3 & 1 & -3 & 1 & -1 \\ 1 & 3 & -1 & -1 & -2 & 10 \end{bmatrix}$$

34. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

35. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 7 & 0 \\ 3 & 1 & 10 & 0 & 23 & 0 \\ 1 & 1 & 4 & 1 & 7 & 0 \\ 1 & -1 & 2 & -2 & 9 & 1 \end{bmatrix}$$

36. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

37. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

38. Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

39. Find $\ker(A)$ for the following matrices.

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$

4.11 ORTHOGONALITY AND THE GRAM SCHMIDT PROCESS

Outcomes

- A. Determine if a given set is orthogonal or orthonormal.
- B. Determine if a given matrix is orthogonal.
- C. Given a linearly independent set, use the Gram-Schmidt Process to find corresponding orthogonal and orthonormal sets.
- D. Find the orthogonal projection of a vector onto a subspace.
- E. Find the least squares approximation for a collection of points.

4.11.1. ORTHOGONAL AND ORTHONORMAL SETS

In this section, we examine what it means for vectors (and sets of vectors) to be orthogonal and orthonormal. First, it is necessary to review some important concepts. You may recall the definitions for the span of a set of vectors and a linear independent set of vectors. We include the definitions and examples here for convenience.

Definition 4.88: Span of a Set of Vectors and Subspace

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is known as the span of these vectors and is written as $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$.

We call a collection of the form $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$ a subspace of \mathbb{R}^n .

Consider the following example.

Example 4.89: Span of Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$.

Solution. You can see that any linear combination of the vectors \vec{u} and \vec{v} yields a vector $\begin{bmatrix} x & y & 0 \end{bmatrix}^T$ in the XY -plane.

Moreover every vector in the XY -plane is in fact such a linear combination of the vectors \vec{u} and \vec{v} . That's because

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (-2x + 3y) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x - y) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Thus $\text{span}\{\vec{u}, \vec{v}\}$ is precisely the XY -plane. \square

The span of a set of vectors in \mathbb{R}^n is what we call a **subspace of \mathbb{R}^n** . A subspace W is characterized by the feature that any linear combination of vectors of W is again a vector contained in W .

Another important property of sets of vectors is called linear independence.

Definition 4.90: Linearly Independent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in \mathbb{R}^n is said to be **linearly independent** if no vector in that set is in the span of the other vectors of that set.

Here is an example.

Example 4.91: Linearly Independent Vectors

Consider vectors $\vec{u} = [1 \ 1 \ 0]^T$, $\vec{v} = [3 \ 2 \ 0]^T$, and $\vec{w} = [4 \ 5 \ 0]^T \in \mathbb{R}^3$. Verify whether the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution. We already verified in Example 4.89 that $\text{span}\{\vec{u}, \vec{v}\}$ is the XY -plane. Since \vec{w} is clearly also in the XY -plane, then the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is **not** linearly independent. \square

In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors. In the previous example you can see that the vector \vec{w} does not help to span any new vector not already in the span of the other two vectors. However you can verify that the set $\{\vec{u}, \vec{v}\}$ is linearly independent, since you will not get the XY -plane as the span of a single vector.

We can also determine if a set of vectors is linearly independent by examining linear combinations. A set of vectors is linearly independent if and only if whenever a linear combination of these vectors equals zero, it follows that all the coefficients equal zero. It is a good exercise to verify this equivalence, and this latter condition is often used as the (equivalent) definition of linear independence.

If a subspace is spanned by a linearly independent set of vectors, then we say that it is a basis for the subspace.

Definition 4.92: Basis of a Subspace

Let V be a subspace of \mathbb{R}^n . Then $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. $\text{span}\{\vec{u}_1, \dots, \vec{u}_k\} = V$
2. $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent

Thus the set of vectors $\{\vec{u}, \vec{v}\}$ from Example 4.91 is a basis for XY -plane in \mathbb{R}^3 since it is both linearly independent and spans the XY -plane.

We can now discuss what is meant by an **orthogonal** set of vectors. We saw in a previous section (see Proposition 4.34) that two vectors \vec{u} and \vec{v} are orthogonal if $\vec{u} \bullet \vec{v} = 0$. This idea can be extended to a set of vectors as follows.

Definition 4.93: Orthogonal Set of Vectors

Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a set of vectors in \mathbb{R}^n . Then this set is called an **orthogonal set** if the following conditions hold:

1. $\vec{u}_i \bullet \vec{u}_j = 0$ for all $i \neq j$
2. $\vec{u}_i \neq \vec{0}$ for all i

If we have an orthogonal set of vectors and normalize each vector so they have length 1, the resulting set is called an **orthonormal set** of vectors. They can be described as follows.

Definition 4.94: Orthonormal Set of Vectors

A set of vectors, $\{\vec{w}_1, \dots, \vec{w}_m\}$ is said to be an **orthonormal set** if

$$\vec{w}_i \bullet \vec{w}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note that all orthonormal sets are orthogonal, but the reverse is not necessarily true since the vectors may not be normalized. In order to normalize the vectors, we simply need divide each by its length.

If an orthogonal set is a basis for a subspace, we call this an orthogonal basis. Similarly, if an orthonormal set is a basis, we call this an orthonormal basis.

Example 4.95: Orthonormal Set

Consider the set of vectors given by

$$\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Show that it is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Solution. One easily verifies that $\vec{u}_1 \bullet \vec{u}_2 = 0$ and $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\vec{u}_1\| = \|\vec{u}_2\| = \sqrt{2} \neq 1$ and thus it is not an orthonormal set.

Thus to find a corresponding orthonormal set, we simply need to normalize each vector. We will write $\{\vec{w}_1, \vec{w}_2\}$ for the corresponding orthonormal set. Then,

$$\begin{aligned} \vec{w}_1 &= \frac{1}{\|\vec{u}_1\|} \vec{u}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} \vec{w}_2 &= \frac{1}{\|\vec{u}_2\|} \vec{u}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Therefore the corresponding orthonormal set is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

You can verify that this set is orthogonal. □

Consider an orthonormal set of vectors in \mathbb{R}^n , written $\{\vec{w}_1, \dots, \vec{w}_k\}$ with $k \leq n$. The span of these vectors is a subspace W of \mathbb{R}^n . If we could show that this orthonormal set is also linearly independent, we would have a basis of W . We will show this in the next theorem.

Theorem 4.96: Orthonormal Basis of a Subspace

Let $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ be an orthonormal set of vectors in \mathbb{R}^n . Then this set is linearly independent hence forms a basis for the subspace $W = \text{span}\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$.

Proof. To show it is a linearly independent set, suppose a linear combination of these vectors equals $\vec{0}$, such as:

$$a_1\vec{w}_1 + a_2\vec{w}_2 + \dots + a_k\vec{w}_k = \vec{0}, a_i \in \mathbb{R}$$

We need to show that all $a_i = 0$. To do so, take the dot product of the vector \vec{w}_1 and the above sum.

$$\begin{aligned}\vec{w}_1 \bullet (a_1\vec{w}_1 + a_2\vec{w}_2 + \dots + a_k\vec{w}_k) &= \vec{w}_1 \bullet \vec{0} \\ a_1(\vec{w}_1 \bullet \vec{w}_1) + a_2(\vec{w}_1 \bullet \vec{w}_2) + \dots + a_k(\vec{w}_1 \bullet \vec{w}_k) &= 0\end{aligned}$$

Now since the set is orthonormal, $\vec{w}_1 \bullet \vec{w}_m = 0$ for all $m \neq 1$, so we have:

$$a_1(\vec{w}_1 \bullet \vec{w}_1) + a_2(0) + \dots + a_k(0) = 0$$

$$a_1\|\vec{w}_1\|^2 = 0$$

Since the set is orthonormal, we know that $\|\vec{w}_1\|^2 = 1$. It follows that $a_1 = 0$.

We can continue in this fashion for the rest of the vectors in the set, and determine that $a_i = 0$ for all $i = 1, 2, \dots, k$. Therefore the set $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ is linearly independent.

Finally since $W = \text{span}\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$, the set of vectors also spans W and therefore forms a basis of W .

□

4.11.2. ORTHOGONAL MATRICES

Recall that the process to find the inverse of a matrix was often cumbersome. In contrast, it was very easy to take the transpose of a matrix. Luckily for some special matrices, the transpose equals the inverse. When an $n \times n$ matrix has all real entries and its transpose equals its inverse, the matrix is called an **orthogonal matrix**.

The precise definition is as follows.

Definition 4.97: Orthogonal Matrices

A real $n \times n$ matrix U is called an **orthogonal matrix** if $UU^T = U^TU = I$.

Note that by Theorem 2.63 it suffices to verify only one of these equalities $UU^T = I$ or $U^TU = I$.

Consider the following example.

Example 4.98: Orthogonal Matrix

Show the matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal.

Solution. All we need to do is verify (one of the equations from) the requirements of Definition 4.97.

$$UU^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $UU^T = I$, this matrix is orthogonal. \square

Here is another example.

Example 4.99: Orthogonal Matrix

Let $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$. Is U orthogonal?

Solution. Again the answer is yes and this can be verified simply by showing that $U^TU = I$:

$$\begin{aligned} U^TU &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

\square

When we say that U is orthogonal, we are saying that

$$\sum_j u_{ij}u_{jk}^T = \sum_j u_{ij}u_{kj} = \delta_{ik}$$

In words, the product of the i^{th} row of U with the k^{th} row gives 1 if $i = k$ and 0 if $i \neq k$. The same is true of the columns because $U^TU = I$ also. Therefore,

$$\sum_j u_{ij}^T u_{jk} = \sum_j u_{ji}u_{jk} = \delta_{ik}$$

which says that the product of one column with another column gives 1 if the two columns are the same and 0 if the two columns are different.

More succinctly, this states that if $\vec{u}_1, \dots, \vec{u}_n$ are the columns of U , an orthogonal matrix, then

$$\vec{u}_i \bullet \vec{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We will say that the columns form an orthonormal set of vectors, and similarly for the rows. Thus a matrix is **orthogonal** if its rows (or columns) form an **orthonormal** set of vectors. Notice that the convention is to call such a matrix orthogonal rather than orthonormal (although this may make more sense!).

Proposition 4.100: Orthonormal Basis

The rows of an $n \times n$ orthogonal matrix form an orthonormal basis of \mathbb{R}^n . Further, any orthonormal basis of \mathbb{R}^n can be used to construct an $n \times n$ orthogonal matrix.

Proof. Recall from Theorem 4.96 that an orthonormal set is linearly independent and forms a basis for its span. Since the rows of an $n \times n$ orthogonal matrix form an orthonormal set, they must be linearly independent. Now we have n linearly independent vectors, and it follows that their span equals \mathbb{R}^n . Therefore these vectors form an orthonormal basis for \mathbb{R}^n .

Suppose we have an orthonormal basis for \mathbb{R}^n . Since the basis will contain n vectors, these can be used to construct an $n \times n$ matrix, with each vector becoming a row. Therefore the matrix is composed of orthonormal rows, which by our above discussion, means that the matrix is orthogonal. \square

Consider the following proposition.

Proposition 4.101: Determinant of Orthogonal Matrices

Suppose U is an orthogonal matrix. Then $\det(U) = \pm 1$.

Proof. This result follows from the properties of determinants. Recall that for any matrix A , $\det(A)^T = \det(A)$. Consider

$$(\det(U))^2 = \det(U^T) \det(U) = \det(U^T U) = \det(I) = 1$$

Therefore $(\det(U))^2 = 1$ and it follows that $\det(U) = \pm 1$. \square

Orthogonal matrices are divided into two classes, proper and improper. The proper orthogonal matrices are those whose determinant equals 1 and the improper ones are those whose determinants equal -1 . The reason for the distinction is that the improper orthogonal matrices are sometimes considered to have no physical significance. These matrices cause a change in orientation which would correspond to material passing through itself in a non physical manner. Thus in considering which coordinate systems must be considered in certain applications, you only need to consider those which are related by a proper orthogonal transformation. Geometrically, the linear transformations determined by the proper orthogonal matrices correspond to the composition of rotations.

4.11.3. GRAM-SCHMIDT PROCESS

The Gram-Schmidt process is an algorithm to transform a set of vectors into an orthonormal set generating the same collection of linear combinations (see Definition 1.32).

The goal of the Gram-Schmidt process is to take a linearly independent set of vectors and transform it into an orthonormal set with the same span. The first objective is to construct an orthogonal set of vectors with the same span, since from there an orthonormal set can be obtained by simply dividing each vector by its length.

Algorithm 4.102: Gram-Schmidt Process

Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be a set of linearly independent vectors in \mathbb{R}^n .

I: Construct a new set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ as follows:

$$\begin{aligned}\vec{v}_1 &= \vec{u}_1 \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \bullet \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ \vec{v}_3 &= \vec{u}_3 - \left(\frac{\vec{u}_3 \bullet \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 - \left(\frac{\vec{u}_3 \bullet \vec{v}_2}{\|\vec{v}_2\|^2} \right) \vec{v}_2 \\ &\vdots \\ \vec{v}_n &= \vec{u}_n - \left(\frac{\vec{u}_n \bullet \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 - \left(\frac{\vec{u}_n \bullet \vec{v}_2}{\|\vec{v}_2\|^2} \right) \vec{v}_2 - \dots - \left(\frac{\vec{u}_n \bullet \vec{v}_{n-1}}{\|\vec{v}_{n-1}\|^2} \right) \vec{v}_{n-1}\end{aligned}$$

II: Now let $\vec{w}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \dots, n$.

Then

1. $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal set.
2. $\{\vec{w}_1, \dots, \vec{w}_n\}$ is an orthonormal set.
3. $\text{span}\{\vec{u}_1, \dots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \text{span}\{\vec{w}_1, \dots, \vec{w}_n\}$.

Proof. The full proof of this algorithm is beyond this course. However to give you an idea that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal set, let

$$a_2 = \frac{\vec{u}_2 \bullet \vec{v}_1}{\|\vec{v}_1\|^2}$$

then:

$$\begin{aligned}\vec{v}_1 \bullet \vec{v}_2 &= \vec{v}_1 \bullet (\vec{u}_2 - a_2 \vec{v}_1) \\ &= \vec{v}_1 \bullet \vec{u}_2 - a_2 (\vec{v}_1 \bullet \vec{v}_1) \\ &= \vec{v}_1 \bullet \vec{u}_2 - \frac{\vec{u}_2 \bullet \vec{v}_1}{\|\vec{v}_1\|^2} \|\vec{v}_1\|^2 \\ &= (\vec{v}_1 \bullet \vec{u}_2) - (\vec{u}_2 \bullet \vec{v}_1) = 0\end{aligned}$$

Now that you have shown that $\{\vec{v}_1, \vec{v}_2\}$ is orthogonal, use the same method as above to show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also orthogonal, and so on.

Then in a similar fashion you show that $\text{span}\{\vec{u}_1, \dots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

Finally defining $\vec{w}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \dots, n$ does not affect orthogonality and yields vectors of length 1, hence an orthonormal set. You can also observe that it does not affect the span either and the proof would be complete. \square

Consider the following example.

Example 4.103: Find Orthonormal Set with Same Span

Consider the set of vectors $\{\vec{u}_1, \vec{u}_2\}$ given as in Example 4.89. That is

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Find an orthonormal set of vectors $\{\vec{w}_1, \vec{w}_2\}$ having the same span.

Solution. We already remarked that the set of vectors in $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so we can proceed with the Gram-Schmidt algorithm:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \bullet \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned}$$

Now simply let

$$\begin{aligned}\vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \vec{w}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\end{aligned}$$

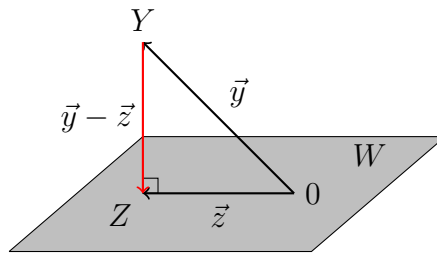
You can verify that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal set of vectors having the same span as $\{\vec{u}_1, \vec{u}_2\}$, namely the XY -plane. \square

4.11.4. ORTHOGONAL PROJECTIONS

An important use of the Gram-Schmidt Process is in orthogonal projections, the focus of this section.

You may recall that a subspace of \mathbb{R}^n is a set of vectors which contains the zero vector, and is closed under addition and scalar multiplication. Let's call such a subspace W . In particular, a plane in \mathbb{R}^n which contains the origin, $(0, 0, \dots, 0)$, is a subspace of \mathbb{R}^n .

Suppose a point Y in \mathbb{R}^n is not contained in W . What point Z in W is closest to Y ? Using the Gram-Schmidt Process, we can find such a point. Let \vec{y}, \vec{z} represent the position vectors of the points Y and Z respectively, with $\vec{y} - \vec{z}$ representing the vector connecting the two points Y and Z . It will follow that if Z is the point on W closest to Y , then $\vec{y} - \vec{z}$ will be perpendicular to W ; in other words, $\vec{y} - \vec{z}$ is orthogonal to W (and to every vector contained in W) as in the following diagram.



The vector \vec{z} is called the **orthogonal projection** of \vec{y} on W . The definition is given as follows.

Definition 4.104: Orthogonal Projection

Let W be a subspace of \mathbb{R}^n , and Y be any point in \mathbb{R}^n . Then the orthogonal projection of Y onto W is given by

$$\vec{z} = \text{proj}_W(\vec{y}) = \left(\frac{\vec{y} \bullet \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 + \left(\frac{\vec{y} \bullet \vec{w}_2}{\|\vec{w}_2\|^2} \right) \vec{w}_2 + \cdots + \left(\frac{\vec{y} \bullet \vec{w}_m}{\|\vec{w}_m\|^2} \right) \vec{w}_m$$

where $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ is any orthogonal basis of W .

Therefore, in order to find the orthogonal projection, we must first find an orthogonal basis for the subspace. Note that one could use an orthonormal basis, but it is not necessary in this case since as you can see above the normalization of each vector is included in the formula for the projection.

Before we explore this further through an example, we show that the orthogonal projection yields a point Z (the point whose position vector is the vector \vec{z} above) which is the point of W closest to Y .

Theorem 4.105: Approximation Theorem

Let W be a subspace of \mathbb{R}^n and Y any point in \mathbb{R}^n . Let Z be the point whose position vector is the orthogonal projection of Y onto W .

Then, Z is the point in W closest to Y .

Proof. To show that Z is the point in W closest to Y , we wish to show that $|\vec{y} - \vec{z}_1| > |\vec{y} - \vec{z}|$ for all $\vec{z}_1 \neq \vec{z} \in W$. We begin by writing $\vec{y} - \vec{z}_1 = (\vec{y} - \vec{z}) + (\vec{z} - \vec{z}_1)$. Now, the vector $\vec{y} - \vec{z}$ is orthogonal to W , and $\vec{z} - \vec{z}_1$ is contained in W . Therefore these vectors are orthogonal to each other. By the Pythagorean Theorem, we have that

$$\|\vec{y} - \vec{z}_1\|^2 = \|\vec{y} - \vec{z}\|^2 + \|\vec{z} - \vec{z}_1\|^2 > \|\vec{y} - \vec{z}\|^2$$

This follows because $\vec{z} \neq \vec{z}_1$ so $\|\vec{z} - \vec{z}_1\|^2 > 0$.

Hence, $\|\vec{y} - \vec{z}_1\|^2 > \|\vec{y} - \vec{z}\|^2$. Taking the square root of each side, we obtain the desired result. \square

Consider the following example.

Example 4.106: Orthogonal Projection

Let W be a plane given by points satisfying $a - 2b + c = 0$. Find the point in W closest to the point $Y = (1, 0, 3)$.

Solution. We must first find an orthogonal basis for W . Notice that W is characterized by all points (a, b, c) where $c = 2b - a$. In other words,

$$W = \left[\begin{array}{c} a \\ b \\ 2b - a \end{array} \right], a, b \in \mathbb{R}$$

We can write W as

$$\begin{aligned} W &= \text{span} \{ \vec{u}_1, \vec{u}_2 \} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \end{aligned}$$

Notice that this span is a basis of W as it is linearly independent. We will use the Gram-Schmidt Process to convert this to an orthogonal basis, $\{\vec{w}_1, \vec{w}_2\}$. In this case, it is only necessary to find an orthogonal basis, and it is not required that it be orthonormal.

$$\vec{w}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \bullet \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{-2}{2} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore an orthogonal basis of W is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We can now use this basis to find the orthogonal projection of the point $Y = (1, 0, 3)$ on the subspace W . We will write the position vector \vec{y} of Y as $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Using Definition

4.104, we continue as follows:

$$\begin{aligned}
 \vec{z} &= \text{proj}_W(\vec{y}) \\
 &= \left(\frac{\vec{y} \bullet \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 + \left(\frac{\vec{y} \bullet \vec{w}_2}{\|\vec{w}_2\|^2} \right) \vec{w}_2 \\
 &= \left(\frac{-2}{2} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{4}{3} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{7}{3} \end{bmatrix}
 \end{aligned}$$

Therefore the point on W closest to the point $(1, 0, 3)$ is $(\frac{1}{3}, \frac{4}{3}, \frac{7}{3})$.

□

Recall that the vector $\vec{y} - \vec{z}$ is perpendicular (orthogonal) to all the vectors contained in the plane W . Using a basis for W , we can in fact find all such vectors which are perpendicular to W . We call this set of vectors the **orthogonal complement** of W and denote it W^\perp .

Definition 4.107: Orthogonal Complement

Let W be a subspace of \mathbb{R}^n . Then the orthogonal complement of W , written W^\perp , is the set of all vectors \vec{x} such that $\vec{x} \bullet \vec{z} = 0$ for all vectors \vec{z} in W .

$$W^\perp = \{ \vec{x} \in \mathbb{R}^n \text{ such that } \vec{x} \bullet \vec{z} = 0 \text{ for all } \vec{z} \in W \}$$

In the next example, we will look at how to find W^\perp .

Example 4.108: Orthogonal Complement

Let W be a plane given by points satisfying $a - 2b + c = 0$. Find the orthogonal complement of W .

Solution.

From Example 4.106 we know that we can write W as

$$W = \text{span} \{ \vec{u}_1, \vec{u}_2 \} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

In order to find W^\perp , we need to find all \vec{x} which are orthogonal to every vector in this span.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. In order to satisfy $\vec{x} \bullet \vec{u}_1 = 0$, the following equation must hold.

$$x_1 - x_3 = 0$$

In order to satisfy $\vec{x} \bullet \vec{u}_2 = 0$, the following equation must hold.

$$x_2 + 2x_3 = 0$$

Both of these equations must be satisfied, so we have the following system of equations.

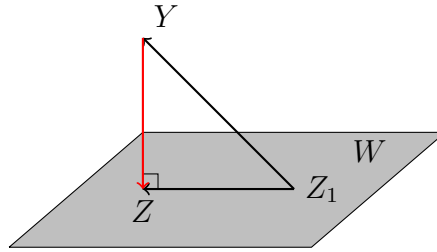
$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

To solve, set up the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

Using Gaussian Elimination, we find that $W^\perp = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

Consider again the following diagram.



Recall that Z is the point in W closest to the point Y . Notice that since the origin is contained in W , the position vector \vec{z} of Z is contained in W . The vector $\vec{y} - \vec{z}$ is in W^\perp . Now, let Z_1 be any other point in W not equal to Z . Then it follows that the distance between Y and Z is shorter than that between Y and Z_1 for all Z_1 . These results are summarized in the following important theorem.

Theorem 4.109: Orthogonal Projection

Let W be a subspace of \mathbb{R}^n , Y be any point in \mathbb{R}^n , and let Z be the point in W closest to Y . Then,

1. The position vector \vec{z} of the point Z is given by $\vec{z} = \text{proj}_W(\vec{y})$
2. $\vec{z} \in W$ and $\vec{y} - \vec{z} \in W^\perp$
3. $|Y - Z| < |Y - Z_1|$ for all $Z_1 \neq Z \in W$

We conclude this section with a final example.

Example 4.110: Vector Written as a Sum of Two Vectors

Let W be a subspace given by $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$, and $Y = (1, 2, 3, 4)$.

Find the point Z in W closest to Y , and moreover write \vec{y} as the sum of a vector in W and a vector in W^\perp .

Solution. From Theorem 4.105, the point Z in W closest to Y is given by $\vec{z} = \text{proj}_W(\vec{y})$.

Notice that since the above vectors already give an orthogonal basis for W , we have:

$$\begin{aligned} \vec{z} &= \text{proj}_W(\vec{y}) \\ &= \left(\frac{\vec{y} \bullet \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 + \left(\frac{\vec{y} \bullet \vec{w}_2}{\|\vec{w}_2\|^2} \right) \vec{w}_2 \\ &= \left(\frac{4}{2} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{10}{5} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

Therefore the point in W closest to Y is $Z = (2, 2, 2, 4)$.

Now, we need to write \vec{y} as the sum of a vector in W and a vector in W^\perp . This can easily be done as follows:

$$\vec{y} = \vec{z} + (\vec{y} - \vec{z})$$

since \vec{z} is in W and as we have seen $\vec{y} - \vec{z}$ is in W^\perp .

The vector $\vec{y} - \vec{z}$ is given by

$$\vec{y} - \vec{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, we can write \vec{y} as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

□

4.11.5. LEAST SQUARES APPROXIMATION

It should not be surprising to hear that many problems do not have a perfect solution, and in these cases the objective is always to try to do the best possible. For example what does one do if there are no solutions to a system of linear equations $A\vec{x} = \vec{b}$? It turns out that what we do is find \vec{x} such that $A\vec{x}$ is as close to \vec{b} as possible. A very important technique that follows from orthogonal projections is that of the least square approximation, and allows us to do exactly that.

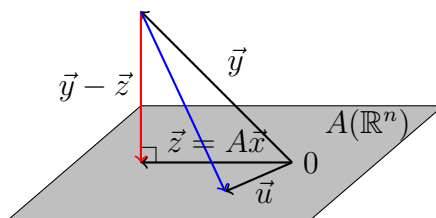
We begin with a lemma.

Lemma 4.111: Matrix Image Subspace

Let A be an $m \times n$ matrix and let $A(\mathbb{R}^n)$ denote the set of vectors in \mathbb{R}^m which are of the form $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. Then $A(\mathbb{R}^n)$ is a subspace of \mathbb{R}^m .

Proof. Let $A\vec{x}$ and $A\vec{y}$ be two vectors of $A(\mathbb{R}^n)$. It suffices to verify that if a, b are scalars, then $aA\vec{x} + bA\vec{y}$ is also in $A(\mathbb{R}^n)$. But $aA\vec{x} + bA\vec{y} = A(a\vec{x} + b\vec{y})$ because A is linear. Since $(a\vec{x} + b\vec{y})$ is a vector in \mathbb{R}^n , it follows that $A(a\vec{x} + b\vec{y})$ is in \mathbb{R}^m as required. \square

The following theorem is a rewording of Theorem 4.109 using the subspace $W = A(\mathbb{R}^n)$ and gives the equivalence of an orthogonality condition with a minimization condition. The following picture illustrates this orthogonality condition and geometric meaning of this theorem.



Theorem 4.112: Existence of Minimizers

Let $\vec{y} \in \mathbb{R}^m$ and let A be an $m \times n$ matrix. Choose $\vec{z} \in W = A(\mathbb{R}^n)$ given by $\vec{z} = \text{proj}_W(\vec{y})$, and let $\vec{x} \in \mathbb{R}^n$ such that $\vec{z} = A\vec{x}$. Then

1. $\vec{y} - A\vec{x} \in W^\perp$
2. $\|\vec{y} - A\vec{x}\| < \|\vec{y} - \vec{u}\|$ for all $\vec{u} \neq \vec{z} \in W$

We note a simple but useful observation.

Lemma 4.113: Transpose and Dot Product

Let A be an $m \times n$ matrix. Then

$$A\vec{x} \bullet \vec{y} = \vec{x} \bullet A^T \vec{y}$$

Proof. This follows from the definitions:

$$A\vec{x} \bullet \vec{y} = \sum_{i,j} a_{ij} x_j y_i = \sum_{i,j} x_j a_{ji} y_i = \vec{x} \bullet A^T \vec{y}$$

□

The next corollary gives the technique of least squares.

Corollary 4.114: Least Squares and Normal Equation

A specific value of \vec{x} which solves the problem of Theorem 4.112 is obtained by solving the equation

$$A^T A \vec{x} = A^T \vec{y}$$

Furthermore, there always exists a solution to this system of equations.

Proof. For \vec{x} the minimizer of Theorem 4.112, $(\vec{y} - A\vec{x}) \bullet A\vec{u} = 0$ for all $\vec{u} \in \mathbb{R}^n$ and from Lemma 4.113, this is the same as saying

$$A^T (\vec{y} - A\vec{x}) \bullet \vec{u} = 0$$

for all $u \in \mathbb{R}^n$. This implies

$$A^T \vec{y} - A^T A \vec{x} = \vec{0}.$$

Therefore, there is a solution to the equation of this corollary, and it solves the minimization problem of Theorem 4.112. □

Note that \vec{x} might not be unique but $A\vec{x}$, the closest point of $A(\mathbb{R}^n)$ to \vec{y} is unique as was shown in the above argument.

An important application of Corollary 4.114 is the problem of finding the least squares regression line in statistics. Suppose you are given points in the xy plane

$$\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$$

and you would like to find constants m and b such that the line $\vec{y} = m\vec{x} + b$ goes through all these points. Of course this will be impossible in general. Therefore, we try to find m, b such that the line will be as close as possible. The desired system is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

which is of the form $\vec{y} = A\vec{x}$. It is desired to choose m and b to make

$$\left\| A \begin{bmatrix} m \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$

as small as possible. According to Theorem 4.112 and Corollary 4.114, the best values for m and b occur as the solution to

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ where } A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$$

Thus, computing $A^T A$,

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}$$

Solving this system of equations for m and b (using Cramer's rule for example) yields:

$$m = \frac{-(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) + (\sum_{i=1}^n x_i y_i) n}{(\sum_{i=1}^n x_i^2) n - (\sum_{i=1}^n x_i)^2}$$

and

$$b = \frac{-(\sum_{i=1}^n x_i) \sum_{i=1}^n x_i y_i + (\sum_{i=1}^n y_i) \sum_{i=1}^n x_i^2}{(\sum_{i=1}^n x_i^2) n - (\sum_{i=1}^n x_i)^2}.$$

Consider the following example.

Example 4.115: Least Squares Regression Line

Find the least squares regression line $\vec{y} = m\vec{x} + b$ for the following set of data points:

$$\{(0, 1), (1, 2), (2, 2), (3, 4), (4, 5)\}$$

Solution. In this case we have $n = 5$ data points and we obtain:

$$\sum_{i=1}^5 x_i = 10 \quad \sum_{i=1}^5 y_i = 14$$

$$\sum_{i=1}^5 x_i y_i = 38 \quad \sum_{i=1}^5 x_i^2 = 30$$

and hence

$$m = \frac{-10 * 14 + 5 * 38}{5 * 30 - 10^2} = 1.00$$

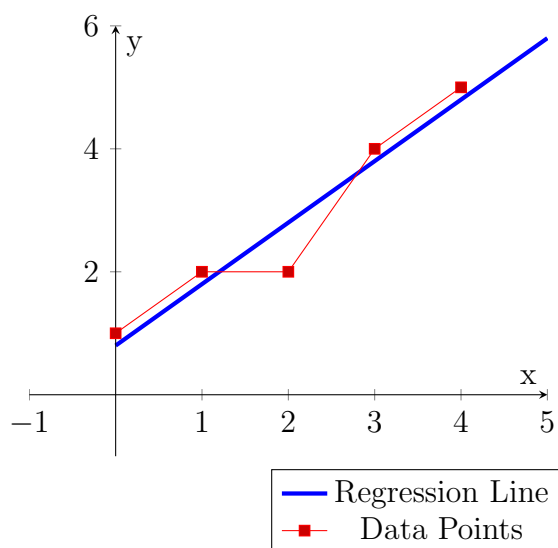
$$b = \frac{-10 * 38 + 14 * 30}{5 * 30 - 10^2} = 0.80$$

The least squares regression line for the set of data points is:

$$\vec{y} = \vec{x} + .8$$

One could use this line to approximate other values for the data. For example for $x = 6$ one could use $y(6) = 6 + .8 = 6.8$ as an approximate value for the data.

The following diagram shows the data points and the corresponding regression line.



□

One could clearly do a least squares fit for curves of the form $y = ax^2 + bx + c$ in the same way. In this case you want to solve as well as possible for a, b , and c the system

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and one would use the same technique as above. Many other similar problems are important, including many in higher dimensions and they are all solved the same way.

4.11.6. EXERCISES

1. Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal.

$$(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{bmatrix}$$

$$(c) \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

2. For U an orthogonal matrix, explain why $\|U\vec{x}\| = \|\vec{x}\|$ for any vector \vec{x} . Next explain why if U is an $n \times n$ matrix with the property that $\|U\vec{x}\| = \|\vec{x}\|$ for all vectors, \vec{x} , then U must be orthogonal. Thus the orthogonal matrices are exactly those which preserve length.
3. Suppose U is an orthogonal $n \times n$ matrix. Explain why $\text{rank}(U) = n$.
4. Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & - & - \\ - & \frac{\sqrt{6}}{3} & - \end{bmatrix}.$$

5. Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & - & - \\ - & 0 & - \end{bmatrix}$$

6. Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & - \\ \frac{2}{3} & 0 & - \\ - & - & \frac{4}{15}\sqrt{5} \end{bmatrix}$$

7. Find an orthonormal basis for the span of each of the following sets of vectors.

$$(a) \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

$$(c) \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

8. Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

9. Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

10. The set $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 3y - z = 0 \right\}$ is a subspace of \mathbb{R}^3 . Find an orthonormal basis for this subspace.

11. Find the least squares solution to the following system.

$$\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 2 \\ 3x + 5y &= 4 \end{aligned}$$

12. You are doing experiments and have obtained the ordered pairs,

$$(0, 1), (1, 2), (2, 3.5), (3, 4)$$

Find m and b such that $\vec{y} = m\vec{x} + b$ approximates these four points as well as possible.

13. Suppose you have several ordered triples, (x_i, y_i, z_i) . Describe how to find a polynomial such as

$$z = a + bx + cy + dxy + ex^2 + fy^2$$

giving the best fit to the given ordered triples.

4.12 APPLICATIONS

Outcomes

A. Apply the concepts of vectors in \mathbb{R}^n to the applications of physics and work.

4.12.1. VECTORS AND PHYSICS

Suppose you push on something. Then, your push is made up of two components, how hard you push and the direction you push. This illustrates the concept of force.

Definition 4.116: Force

Force is a vector. The magnitude of this vector is a measure of how hard it is pushing. It is measured in units such as Newtons or pounds or tons. The direction of this vector is the direction in which the push is taking place.

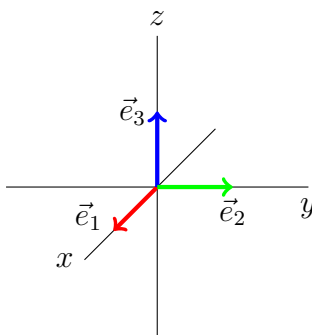
Vectors are used to model force and other physical vectors like velocity. As with all vectors, a vector modeling force has two essential ingredients, its magnitude and its direction.

Recall the special vectors which point along the coordinate axes. These are given by

$$\vec{e}_i = [0 \cdots 0 \ 1 \ 0 \cdots 0]^T$$

where the 1 is in the i^{th} slot and there are zeros in all the other spaces. The direction of \vec{e}_i is referred to as the i^{th} direction.

Consider the following picture which illustrates the case of \mathbb{R}^3 . Recall that in \mathbb{R}^3 , we may refer to these vectors as \vec{i}, \vec{j} , and \vec{k} .



Given a vector $\vec{u} = [u_1 \cdots u_n]^T$, it follows that

$$\vec{u} = u_1 \vec{e}_1 + \cdots + u_n \vec{e}_n = \sum_{k=1}^n u_k \vec{e}_k$$

What does addition of vectors mean physically? Suppose two forces are applied to some object. Each of these would be represented by a force vector and the two forces acting together would yield an overall force acting on the object which would also be a force vector known as the resultant. Suppose the two vectors are $\vec{u} = \sum_{k=1}^n u_k \vec{e}_k$ and $\vec{v} = \sum_{k=1}^n v_k \vec{e}_k$. Then the vector \vec{u} involves a component in the i^{th} direction given by $u_i \vec{e}_i$, while the component in the i^{th} direction of \vec{v} is $v_i \vec{e}_i$. Then the vector $\vec{u} + \vec{v}$ should have a component in the i^{th}

direction equal to $(u_i + v_i) \vec{e}_i$. This is exactly what is obtained when the vectors, \vec{u} and \vec{v} are added.

$$\begin{aligned}\vec{u} + \vec{v} &= [u_1 + v_1 \cdots u_n + v_n]^T \\ &= \sum_{i=1}^n (u_i + v_i) \vec{e}_i\end{aligned}$$

Thus the addition of vectors according to the rules of addition in \mathbb{R}^n which were presented earlier, yields the appropriate vector which duplicates the cumulative effect of all the vectors in the sum.

Consider now some examples of vector addition.

Example 4.117: The Resultant of Three Forces

There are three ropes attached to a car and three people pull on these ropes. The first exerts a force of $\vec{F}_1 = 2\vec{i} + 3\vec{j} - 2\vec{k}$ Newtons, the second exerts a force of $\vec{F}_2 = 3\vec{i} + 5\vec{j} + \vec{k}$ Newtons and the third exerts a force of $5\vec{i} - \vec{j} + 2\vec{k}$ Newtons. Find the total force in the direction of \vec{i} .

Solution. To find the total force, we add the vectors as described above. This is given by

$$\begin{aligned}& (2\vec{i} + 3\vec{j} - 2\vec{k}) + (3\vec{i} + 5\vec{j} + \vec{k}) + (5\vec{i} - \vec{j} + 2\vec{k}) \\ &= (2 + 3 + 5)\vec{i} + (3 + 5 + -1)\vec{j} + (-2 + 1 + 2)\vec{k} \\ &= 10\vec{i} + 7\vec{j} + \vec{k}\end{aligned}$$

Hence, the total force is $10\vec{i} + 7\vec{j} + \vec{k}$ Newtons. Therefore, the force in the \vec{i} direction is 10 Newtons. \square

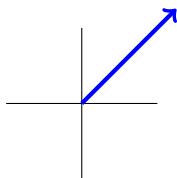
Consider another example.

Example 4.118: Finding a Vector from Geometric Description

An airplane flies North East at 100 miles per hour. Write this as a vector.

Solution.

A picture of this situation follows.



Therefore, we need to find the vector \vec{u} which has length 100 and direction as shown in this diagram. We can consider the vector \vec{u} as the hypotenuse of a right triangle having

equal sides, since the direction of \vec{u} corresponds with the 45° line. The sides, corresponding to the \vec{i} and \vec{j} directions, should be each of length $100/\sqrt{2}$. Therefore, the vector is given by

$$\vec{u} = \frac{100}{\sqrt{2}}\vec{i} + \frac{100}{\sqrt{2}}\vec{j} = \left[\frac{100}{\sqrt{2}} \quad \frac{100}{\sqrt{2}} \right]^T$$

□

This example also motivates the concept of **velocity**, defined below.

Definition 4.119: Speed and Velocity

The **speed** of an object is a measure of how fast it is going. It is measured in units of length per unit time. For example, miles per hour, kilometers per minute, feet per second. The **velocity** is a vector having the speed as the magnitude but also specifying the direction.

Thus the velocity vector in the above example is $\frac{100}{\sqrt{2}}\vec{i} + \frac{100}{\sqrt{2}}\vec{j}$, while the speed is 100 miles per hour.

Consider the following example.

Example 4.120: Position From Velocity and Time

The velocity of an airplane is $100\vec{i} + \vec{j} + \vec{k}$ measured in kilometers per hour and at a certain instant of time its position is $(1, 2, 1)$. Find the position of this airplane one minute later.

Solution. Here imagine a Cartesian coordinate system in which the third component is altitude and the first and second components are measured on a line from West to East and a line from South to North.

Consider the vector $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$, which is the initial position vector of the airplane. As the plane moves, the position vector changes according to the velocity vector. After one minute (considered as $\frac{1}{60}$ of an hour) the airplane has moved in the \vec{i} direction a distance of $100 \times \frac{1}{60} = \frac{5}{3}$ kilometer. In the \vec{j} direction it has moved $\frac{1}{60}$ kilometer during this same time, while it moves $\frac{1}{60}$ kilometer in the \vec{k} direction. Therefore, the new displacement vector for the airplane is

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T + \begin{bmatrix} \frac{5}{3} & \frac{1}{60} & \frac{1}{60} \end{bmatrix}^T = \begin{bmatrix} \frac{8}{3} & \frac{121}{60} & \frac{121}{60} \end{bmatrix}^T$$

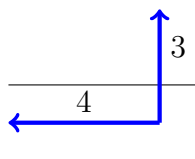
□

Now consider an example which involves combining two velocities.

Example 4.121: Sum of Two Velocities

A certain river is one half kilometer wide with a current flowing at 4 kilometers per hour from East to West. A man swims directly toward the opposite shore from the South bank of the river at a speed of 3 kilometers per hour. How far down the river does he find himself when he has swam across? How far does he end up swimming?

Solution. Consider the following picture which demonstrates the above scenario.



First we want to know the total time of the swim across the river. The velocity in the direction across the river is 3 kilometers per hour, and the river is $\frac{1}{2}$ kilometer wide. It follows the trip takes $1/6$ hour or 10 minutes.

Now, we can compute how far downstream he will end up. Since the river runs at a rate of 4 kilometers per hour, and the trip takes $1/6$ hour, the distance traveled downstream is given by $4 \left(\frac{1}{6}\right) = \frac{2}{3}$ kilometers.

The distance traveled by the swimmer is given by the hypotenuse of a right triangle. The two arms of the triangle are given by the distance across the river, $\frac{1}{2}$ km, and the distance traveled downstream, $\frac{2}{3}$ km. Then, using the Pythagorean Theorem, we can calculate the total distance d traveled.

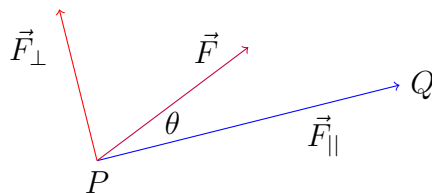
$$d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{5}{6} \text{ km}$$

Therefore, the swimmer travels a total distance of $\frac{5}{6}$ kilometers. \square

4.12.2. WORK

The mathematical concept of work is an application of vectors in \mathbb{R}^n . The physical concept of work differs from the notion of work employed in ordinary conversation. For example, suppose you were to slide a 150 pound weight off a table which is three feet high and shuffle along the floor for 50 yards, keeping the height always three feet and then deposit this weight on another three foot high table. The physical concept of work would indicate that the force exerted by your arms did no work during this project. The reason for this definition is that even though your arms exerted considerable force on the weight, the direction of motion was at right angles to the force they exerted. The only part of a force which does work in the sense of physics is the component of the force in the direction of motion.

Work is defined to be the magnitude of the component of this force times the distance over which it acts, when the component of force points in the direction of motion. In the case where the force points in exactly the opposite direction of motion work is given by (-1) times the magnitude of this component times the distance. Thus the work done by a force on an object as the object moves from one point to another is a measure of the extent to which the force contributes to the motion. This is illustrated in the following picture in the case where the given force contributes to the motion.



Recall that for any vector \vec{u} in \mathbb{R}^n , we can write \vec{u} as a sum of two vectors, as in

$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$$

For any force \vec{F} , we can write this force as the sum of a vector in the direction of the motion and a vector perpendicular to the motion. In other words,

$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

In the above picture the force, \vec{F} is applied to an object which moves on the straight line from P to Q . There are two vectors shown, \vec{F}_{\parallel} and \vec{F}_{\perp} and the picture is intended to indicate that when you add these two vectors you get \vec{F} . In other words, $\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}$. Notice that \vec{F}_{\parallel} acts in the direction of motion and \vec{F}_{\perp} acts perpendicular to the direction of motion. Only \vec{F}_{\parallel} contributes to the work done by \vec{F} on the object as it moves from P to Q . \vec{F}_{\parallel} is called the component of the force in the direction of motion. From trigonometry, you see the magnitude of \vec{F}_{\parallel} should equal $\|\vec{F}\| |\cos \theta|$. Thus, since \vec{F}_{\parallel} points in the direction of the vector from P to Q , the total work done should equal

$$\|\vec{F}\| \|\overrightarrow{PQ}\| \cos \theta = \|\vec{F}\| \|\vec{q} - \vec{p}\| \cos \theta$$

Now, suppose the included angle had been obtuse. Then the work done by the force \vec{F} on the object would have been negative because \vec{F}_{\parallel} would point in -1 times the direction of the motion. In this case, $\cos \theta$ would also be negative and so it is still the case that the work done would be given by the above formula. Thus from the geometric description of the dot product given above, the work equals

$$\|\vec{F}\| \|\vec{q} - \vec{p}\| \cos \theta = \vec{F} \bullet (\vec{q} - \vec{p})$$

This explains the following definition.

Definition 4.122: Work Done on an Object by a Force

Let \vec{F} be a force acting on an object which moves from the point P to the point Q , which have position vectors given by \vec{p} and \vec{q} respectively. Then the **work** done on the object by the given force equals $\vec{F} \bullet (\vec{q} - \vec{p})$.

Consider the following example.

Example 4.123: Finding Work

Let $\vec{F} = [2 \ 7 \ -3]^T$ Newtons. Find the work done by this force in moving from the point $(1, 2, 3)$ to the point $(-9, -3, 4)$ along the straight line segment joining these points where distances are measured in meters.

Solution. First, compute the vector $\vec{q} - \vec{p}$, given by

$$[-9 \ -3 \ 4]^T - [1 \ 2 \ 3]^T = [-10 \ -5 \ 1]^T$$

According to Definition 4.122 the work done is

$$\begin{aligned} [2 \ 7 \ 3]^T \bullet [-10 \ -5 \ 1]^T &= -20 + (-35) + (-3) \\ &= -58 \text{ Newton meters} \end{aligned}$$

□

Note that if the force had been given in pounds and the distance had been given in feet, the units on the work would have been foot pounds. In general, work has units equal to units of a force times units of a length. Recall that 1 Newton meter is equal to 1 Joule. Also notice that the work done by the force can be negative as in the above example.

4.12.3. EXERCISES

1. The wind blows from the South at 20 kilometers per hour and an airplane which flies at 600 kilometers per hour in still air is heading East. Find the velocity of the airplane and its location after two hours.
2. The wind blows from the West at 30 kilometers per hour and an airplane which flies at 400 kilometers per hour in still air is heading North East. Find the velocity of the airplane and its position after two hours.
3. The wind blows from the North at 10 kilometers per hour. An airplane which flies at 300 kilometers per hour in still air is supposed to go to the point whose coordinates are at $(100, 100)$. In what direction should the airplane fly?

4. Three forces act on an object. Two are $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ Newtons. Find the third force if the object is not to move.

5. Three forces act on an object. Two are $\begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ Newtons. Find the third force if the total force on the object is to be $\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$.

6. A river flows West at the rate of b miles per hour. A boat can move at the rate of 8 miles per hour. Find the smallest value of b such that it is not possible for the boat to proceed directly across the river.
7. The wind blows from West to East at a speed of 50 miles per hour and an airplane which travels at 400 miles per hour in still air is heading North West. What is the velocity of the airplane relative to the ground? What is the component of this velocity in the direction North?
8. The wind blows from West to East at a speed of 60 miles per hour and an airplane can travel travels at 100 miles per hour in still air. How many degrees West of North should the airplane head in order to travel exactly North?
9. The wind blows from West to East at a speed of 50 miles per hour and an airplane which travels at 400 miles per hour in still air heading somewhat West of North so that, with the wind, it is flying due North. It uses 30.0 gallons of gas every hour. If it has to travel 600.0 miles due North, how much gas will it use in flying to its destination?
10. An airplane is flying due north at 500.0 miles per hour but it is not actually going due North because there is a wind which is pushing the airplane due east at 40.0 miles per hour. After one hour, the plane starts flying 30° East of North. Assuming the plane starts at $(0, 0)$, where is it after 2 hours? Let North be the direction of the positive y axis and let East be the direction of the positive x axis.
11. City A is located at the origin $(0, 0)$ while city B is located at $(300, 500)$ where distances are in miles. An airplane flies at 250 miles per hour in still air. This airplane wants to fly from city A to city B but the wind is blowing in the direction of the positive y axis at a speed of 50 miles per hour. Find a unit vector such that if the plane heads in this direction, it will end up at city B having flown the shortest possible distance. How long will it take to get there?
12. A certain river is one half mile wide with a current flowing at 3.0 miles per hour from East to West. A man takes a boat directly toward the opposite shore from the South bank of the river at a speed of 5.0 miles per hour. How far down the river does he find himself when he has swam across? How far does he end up traveling?
13. A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man can swim at 3 miles per hour in still water. In what direction should he swim in order to travel directly across the river? What would the answer to this problem be if the river flowed at 3 miles per hour and the man could swim only at the rate of 2 miles per hour?
14. Three forces are applied to a point which does not move. Two of the forces are $2\vec{i} + 2\vec{j} - 6\vec{k}$ Newtons and $8\vec{i} + 8\vec{j} + 3\vec{k}$ Newtons. Find the third force.

15. The total force acting on an object is to be $4\vec{i} + 2\vec{j} - 3\vec{k}$ Newtons. A force of $-3\vec{i} - 1\vec{j} + 8\vec{k}$ Newtons is being applied. What other force should be applied to achieve the desired total force?
16. A bird flies from its nest 8 km in the direction $\frac{5}{6}\pi$ north of east where it stops to rest on a tree. It then flies 1 km in the direction due southeast and lands atop a telephone pole. Place an xy coordinate system so that the origin is the bird's nest, and the positive x axis points east and the positive y axis points north. Find the displacement vector from the nest to the telephone pole.
17. If \vec{F} is a force and \vec{D} is a vector, show $\text{proj}_{\vec{D}}(\vec{F}) = (\|\vec{F}\| \cos \theta) \vec{u}$ where \vec{u} is the unit vector in the direction of \vec{D} , where $\vec{u} = \vec{D}/\|\vec{D}\|$ and θ is the included angle between the two vectors, \vec{F} and \vec{D} . $\|\vec{F}\| \cos \theta$ is sometimes called the component of the force, \vec{F} in the direction, \vec{D} .
18. A boy drags a sled for 100 feet along the ground by pulling on a rope which is 20 degrees from the horizontal with a force of 40 pounds. How much work does this force do?
19. A girl drags a sled for 200 feet along the ground by pulling on a rope which is 30 degrees from the horizontal with a force of 20 pounds. How much work does this force do?
20. A large dog drags a sled for 300 feet along the ground by pulling on a rope which is 45 degrees from the horizontal with a force of 20 pounds. How much work does this force do?
21. How much work does it take to slide a crate 20 meters along a loading dock by pulling on it with a 200 Newton force at an angle of 30° from the horizontal? Express your answer in Newton meters.
22. An object moves 10 meters in the direction of \vec{j} . There are two forces acting on this object, $\vec{F}_1 = \vec{i} + \vec{j} + 2\vec{k}$, and $\vec{F}_2 = -5\vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?
23. An object moves 10 meters in the direction of $\vec{j} + \vec{i}$. There are two forces acting on this object, $\vec{F}_1 = \vec{i} + 2\vec{j} + 2\vec{k}$, and $\vec{F}_2 = 5\vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?
24. An object moves 20 meters in the direction of $\vec{k} + \vec{j}$. There are two forces acting on this object, $\vec{F}_1 = \vec{i} + \vec{j} + 2\vec{k}$, and $\vec{F}_2 = \vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force.

5. LINEAR TRANSFORMATIONS

5.1 LINEAR TRANSFORMATIONS

Outcomes

- A. Understand the definition of a linear transformation, and that all linear transformations are determined by matrix multiplication.

Recall that when we multiply an $m \times n$ matrix by an $n \times 1$ column vector, the result is an $m \times 1$ column vector. In this section we will discuss how, through matrix multiplication, an $m \times n$ matrix **transforms** an $n \times 1$ column vector into an $m \times 1$ column vector.

Recall that the $n \times 1$ vector given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to belong to \mathbb{R}^n , which is the set of all $n \times 1$ vectors. In this section, we will discuss transformations of vectors in \mathbb{R}^n .

Consider the following example.

Example 5.1: A Function Which Transforms Vectors

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Show that by matrix multiplication A transforms vectors in \mathbb{R}^3 into vectors in \mathbb{R}^2 .

Solution. First, recall that vectors in \mathbb{R}^3 are vectors of size 3×1 , while vectors in \mathbb{R}^2 are of size 2×1 . If we multiply A , which is a 2×3 matrix, by a 3×1 vector, the result will be a 2×1 vector. This is what we mean when we say that A *transforms* vectors.

Now, for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in \mathbb{R}^3 , multiply on the left by the given matrix to obtain the new vector.

This product looks like

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

The resulting product is a 2×1 vector which is determined by the choice of x and y . Here are some numerical examples.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in \mathbb{R}^3 was transformed by the matrix into the vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in \mathbb{R}^2 .

Here is another example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

□

The idea is to define a function which takes vectors in \mathbb{R}^3 and delivers new vectors in \mathbb{R}^2 . In this case, that function is multiplication by the matrix A .

Let T denote such a function. The notation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ means that the function T transforms vectors in \mathbb{R}^n into vectors in \mathbb{R}^m . The notation $T(\vec{x})$ means the transformation T applied to the vector \vec{x} . The above example demonstrated a transformation achieved by matrix multiplication. In this case, we often write

$$T_A(\vec{x}) = A\vec{x}$$

Therefore, T_A is the transformation determined by the matrix A . In this case we say that T is a matrix transformation.

Recall the properties of matrix multiplication. The pertinent property here is 2.6 which states that for k and p scalars,

$$A(kB + pC) = kAB + pAC$$

In particular, for A an $m \times n$ matrix and B and C , $n \times 1$ vectors in \mathbb{R}^n , this formula holds.

In other words, this means that matrix multiplication gives an example of a linear transformation, which we will now define.

Definition 5.2: Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function, where for each $\vec{x} \in \mathbb{R}^n$, $T(\vec{x}) \in \mathbb{R}^m$. Then T is a **linear transformation** if whenever k, p are scalars and \vec{x}_1 and \vec{x}_2 are vectors in \mathbb{R}^n ($n \times 1$ vectors),

$$T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$$

We began this section by discussing matrix transformations, where multiplication by a matrix transforms vectors. These matrix transformations are in fact linear transformations.

Theorem 5.3: Matrix Transformations are Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a transformation defined by $T(\vec{x}) = A\vec{x}$. Then T is a linear transformation.

It turns out that every linear transformation can be expressed as a matrix transformation, and thus linear transformations are exactly the same as matrix transformations. This will be the content of the next section.

5.1.1. EXERCISES

1. Show the map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ where A is an $m \times n$ matrix and \vec{x} is an $n \times 1$ column vector is a linear transformation.
2. Show that the function $T_{\vec{v}}$ defined by $T_{\vec{v}}(\vec{w}) = \vec{w} - \text{proj}_{\vec{v}}(\vec{w})$ is also a linear transformation.
3. Let \vec{u} be a fixed vector. The function $T_{\vec{u}}$ defined by $T_{\vec{u}}\vec{v} = \vec{u} + \vec{v}$ has the effect of translating all vectors by adding $\vec{u} \neq \vec{0}$. Show this is not a linear transformation. Explain why it is not possible to represent $T_{\vec{u}}$ in \mathbb{R}^3 by multiplying by a 3×3 matrix.

5.2 THE MATRIX OF A LINEAR TRANSFORMATION

Outcomes

- A. Find the matrix of a linear transformation and determine the action on a vector in \mathbb{R}^n .

In the above examples, the action of the linear transformations was to multiply by a matrix. It turns out that this is always the case for linear transformations. If T is **any** linear transformation which maps \mathbb{R}^n to \mathbb{R}^m , there is **always** an $m \times n$ matrix A with the property that

$$T(\vec{x}) = A\vec{x} \tag{5.1}$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 5.4: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then we can find a matrix A such that $T(\vec{x}) = A\vec{x}$. In this case, we say that T is determined or induced by the matrix A .

Here is why. Suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and you want to find the matrix defined by this linear transformation as described in 5.1. Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^n x_i \vec{e}_i$$

where \vec{e}_i is the i^{th} column of I_n , that is the $n \times 1$ vector which has zeros in every slot but the i^{th} and a 1 in this slot.

Then since T is linear,

$$\begin{aligned} T(\vec{x}) &= \sum_{i=1}^n x_i T(\vec{e}_i) \\ &= \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

Therefore, the desired matrix is obtained from constructing the i^{th} column as $T(\vec{e}_i)$. We state this formally as the following theorem.

Theorem 5.5: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then the matrix A satisfying $T(\vec{x}) = A\vec{x}$ is given by

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

where \vec{e}_i is the i^{th} column of I_n , and then $T(\vec{e}_i)$ is the i^{th} column of A .

The following Corollary is an essential result.

Corollary 5.6: Matrix and Linear Transformation

A transformation T is a linear transformation if and only if it is a matrix transformation.

Consider the following example.

Example 5.7: The Matrix of a Linear Transformation

Suppose T is a linear transformation, $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all \vec{x} .

Solution. By Theorem 5.5 we construct A as follows:

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

In this case, A will be a 2×3 matrix, so we need to find $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A . Hence,

$$A = \begin{bmatrix} 1 & 9 & 1 \\ 2 & -3 & 1 \end{bmatrix}$$

□

In this example, we were given the resulting vectors of $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$. Constructing the matrix A was simple, as we could simply use these vectors as the columns of A .

The next example shows how to find A when we are not given the $T(\vec{e}_i)$ so clearly.

Example 5.8: The Matrix of Linear Transformation: Inconveniently Defined

Suppose T is known to be a linear transformation, $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the matrix A of the transformation T .

Solution. By Theorem 5.5 to find this matrix, we need to determine the action of T on \vec{e}_1 and \vec{e}_2 . In Example 5.7, we were given these resulting vectors. However, in this example, we have been given T of two different vectors. How can we find out the action of T on \vec{e}_1 and \vec{e}_2 ? In particular for \vec{e}_1 , suppose there exist x and y such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{5.2}$$

Then, since T is linear,

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = xT \begin{bmatrix} 1 \\ 1 \end{bmatrix} + yT \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Substituting in values, this sum becomes

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad (5.3)$$

Therefore, if we know the values of x and y which satisfy 5.2, we can substitute these into equation 5.3. By doing so, we find $T(\vec{e}_1)$ which is the first column of the matrix A .

We proceed to find x and y . We do so by solving 5.2, which can be done by solving the system

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

We see that $x = 1$ and $y = 1$ is the solution to this system. Substituting these values into equation 5.3, we have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Therefore $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is the first column of A .

Computing the second column is done in the same way, and is left as an exercise.

The resulting matrix A is given by

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$

□

This example illustrates a very long procedure for finding the matrix of A . While this method is reliable and will always result in the correct matrix A , the following procedure provides an alternative method.

Recall that

$$\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$$

denotes a matrix which has for its i^{th} column the vector A_i .

Procedure 5.9: Finding the Matrix of Inconveniently Defined Linear Transformation

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation. Suppose there exist vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ in \mathbb{R}^n such that $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$ exists, and

$$T(\vec{a}_i) = \vec{b}_i$$

Then the matrix of T must be of the form

$$\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$$

We will illustrate this procedure in the following example. You may also find it useful to work through Example 5.8 using this procedure.

**Example 5.10: Matrix of a Linear Transformation
Given Inconveniently**

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation and

$$T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the matrix of this linear transformation.

Solution. By Procedure 5.9, $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$ and $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Then, Procedure 5.9 claims that the matrix of T is

$$C = BA^{-1} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

Indeed you can first verify that $T(\vec{x}) = C\vec{x}$ for the 3 vectors above:

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But more generally $T(\vec{x}) = C\vec{x}$ for any \vec{x} . To see this, let $\vec{y} = A^{-1}\vec{x}$ and then using linearity of T :

$$T(\vec{x}) = T(A\vec{y}) = T\left(\sum_i \vec{y}_i \vec{a}_i\right) = \sum \vec{y}_i T(\vec{a}_i) = \sum \vec{y}_i \vec{b}_i = B\vec{y} = BA^{-1}\vec{x} = C\vec{x}$$

□

Recall the dot product discussed earlier. Consider the map $\vec{v} \mapsto \text{proj}_{\vec{u}}(\vec{v})$. It turns out that this map is linear, a result which follows from the properties of the dot product. This is shown as follows.

$$\begin{aligned} \text{proj}_{\vec{u}}(k\vec{v} + p\vec{w}) &= \left(\frac{(k\vec{v} + p\vec{w}) \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u} \\ &= k \left(\frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u} + p \left(\frac{\vec{w} \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u} \\ &= k \text{proj}_{\vec{u}}(\vec{v}) + p \text{proj}_{\vec{u}}(\vec{w}) \end{aligned}$$

Consider the following example.

Example 5.11: Matrix of a Projection Map

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and let T be the projection map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by $T(\vec{v}) = \text{proj}_{\vec{u}}(\vec{v})$ for any $\vec{v} \in \mathbb{R}^3$.

1. Does this transformation come from multiplication by a matrix?
2. If so, what is the matrix?

Solution.

1. First, we have just seen that $T(\vec{v}) = \text{proj}_{\vec{u}}(\vec{v})$ is linear. Therefore by Theorem 5.4, we can find a matrix A such that $T(\vec{x}) = A\vec{x}$.
2. The columns of the matrix for T are defined above as $T(\vec{e}_i)$. It follows that $T(\vec{e}_i) = \text{proj}_{\vec{u}}(\vec{e}_i)$ gives the i^{th} column of the desired matrix. Therefore, we need to find

$$\text{proj}_{\vec{u}}(\vec{e}_i) = \left(\frac{\vec{e}_i \bullet \vec{u}}{\vec{u} \bullet \vec{u}} \right) \vec{u}$$

For the given vector \vec{u} , this implies the columns of the desired matrix are

$$\frac{1}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{2}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{3}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which you can verify using Definition 4.37. Hence the matrix of T is

$$\frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

□

5.2.1. EXERCISES

1. Consider the following functions which map \mathbb{R}^n to \mathbb{R}^n .
 - (a) T multiplies the j^{th} component of \vec{x} by a nonzero number b .
 - (b) T replaces the i^{th} component of \vec{x} with b times the j^{th} component added to the i^{th} component.

(c) T switches the i^{th} and j^{th} components.

Show these functions are linear transformations and describe their matrices A such that $T(\vec{x}) = A\vec{x}$.

2. You are given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and you know that

$$T(A_i) = B_i$$

where $[A_1 \ \cdots \ A_n]^{-1}$ exists. Show that the matrix of T is of the form

$$[B_1 \ \cdots \ B_n] [A_1 \ \cdots \ A_n]^{-1}$$

3. Suppose T is a linear transformation such that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} &= \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ T \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \\ T \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \end{aligned}$$

Find the matrix of T . That is find A such that $T(\vec{x}) = A\vec{x}$.

4. Suppose T is a linear transformation such that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ T \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} &= \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \\ T \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} &= \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

Find the matrix of T . That is find A such that $T(\vec{x}) = A\vec{x}$.

5. Suppose T is a linear transformation such that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} &= \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \\ T \begin{bmatrix} -1 \\ -2 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \\ T \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 5 \\ 3 \\ -3 \end{bmatrix} \end{aligned}$$

Find the matrix of T . That is find A such that $T(\vec{x}) = A\vec{x}$.

6. Suppose T is a linear transformation such that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \\ T \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ T \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \end{aligned}$$

Find the matrix of T . That is find A such that $T(\vec{x}) = A\vec{x}$.

7. Suppose T is a linear transformation such that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 2 \\ -18 \end{bmatrix} &= \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} \\ T \begin{bmatrix} -1 \\ -1 \\ 15 \end{bmatrix} &= \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \\ T \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} &= \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} \end{aligned}$$

Find the matrix of T . That is find A such that $T(\vec{x}) = A\vec{x}$.

8. Consider the following functions $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Show that each is a linear transformation and determine for each the matrix A such that $T(\vec{x}) = A\vec{x}$.

$$\begin{aligned} \text{(a)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + 2y + 3z \\ 2y - 3x + z \end{bmatrix} \\ \text{(b)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 7x + 2y + z \\ 3x - 11y + 2z \end{bmatrix} \\ \text{(c)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3x + 2y + z \\ x + 2y + 6z \end{bmatrix} \\ \text{(d)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2y - 5x + z \\ x + y + z \end{bmatrix} \end{aligned}$$

9. Consider the following functions $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Explain why each of these functions T is not linear.

$$\begin{aligned} \text{(a)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + 2y + 3z + 1 \\ 2y - 3x + z \end{bmatrix} \\ \text{(b)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + 2y^2 + 3z \\ 2y + 3x + z \end{bmatrix} \\ \text{(c)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} \sin x + 2y + 3z \\ 2y + 3x + z \end{bmatrix} \\ \text{(d)} \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + 2y + 3z \\ 2y + 3x - \ln z \end{bmatrix} \end{aligned}$$

10. Suppose

$$\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}^{-1}$$

exists where each $A_j \in \mathbb{R}^n$ and let vectors $\{B_1, \dots, B_n\}$ in \mathbb{R}^m be given. Show that there **always** exists a linear transformation T such that $T(A_i) = B_i$.

11. Find the matrix for $T(\vec{w}) = \text{proj}_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$.
12. Find the matrix for $T(\vec{w}) = \text{proj}_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix}^T$.
13. Find the matrix for $T(\vec{w}) = \text{proj}_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^T$.

5.3 PROPERTIES OF LINEAR TRANSFORMATIONS

Outcomes

A. Find the composite of transformations and the inverse of a transformation.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there are some important properties of T which will be examined in this section. Consider the following theorem.

Theorem 5.12: Properties of Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

- $T(0\vec{x}) = 0T(\vec{x})$. Hence $T(\vec{0}) = \vec{0}$. (T preserves the zero vector)
- $T((-1)\vec{x}) = (-1)T(\vec{x})$. Hence $T(-\vec{x}) = -T(\vec{x})$. (T preserves the negative of a vector)
- Let $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$ and $a_1, \dots, a_k \in \mathbb{R}$. Then if $\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_k\vec{x}_k$, it follows that $T(\vec{y}) = T(a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_k\vec{x}_k) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \dots + a_kT(\vec{x}_k)$. (T preserves linear combinations).

Consider the following example.

Example 5.13: Linear Combination

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be a linear transformation such that

$$T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$.

Solution. Using the third property in Theorem 5.12, we can find $T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ by writing

$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

Therefore we want to find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

The necessary augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array} \right]$$

Hence $a = 1, b = -2$ and

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Now, using the third property above, we have

$$\begin{aligned} T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T \left(1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right) \\ &= T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix} \end{aligned}$$

$$\text{Therefore, } T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}.$$

□

Suppose two linear transformations act on the same vector \vec{x} , first the transformation T and then a second transformation given by S . We can find the **composite** transformation that results from applying both transformations.

Definition 5.14: Composition of Linear Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Then the **composite** of S and T is

$$S \circ T : \mathbb{R}^k \mapsto \mathbb{R}^m$$

The action of $S \circ T$ is given by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k$$

Notice that the resulting vector will be in \mathbb{R}^m . Be careful to observe the order of transformations. We write $S \circ T$ but apply the transformation T first, followed by S .

Theorem 5.15: Composition of Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations such that T is induced by the matrix A and S is induced by the matrix B . Then $S \circ T$ is a linear transformation which is induced by the matrix BA .

Consider the following example.

Example 5.16: Composition of Transformations

Let T be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

and S a linear transformation induced by the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the matrix of the composite transformation $S \circ T$. Then, find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution. By Theorem 5.15, the matrix of $S \circ T$ is given by BA .

$$BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix}$$

To find $(S \circ T)(\vec{x})$, multiply \vec{x} by BA as follows

$$\begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

To check, first determine $T(\vec{x})$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Then, compute $S(T(\vec{x}))$ as follows:

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

□

Consider a composite transformation $S \circ T$, and suppose that this transformation acted such that $(S \circ T)(\vec{x}) = \vec{x}$. That is, the transformation S took the vector $T(\vec{x})$ and returned it to \vec{x} . In this case, S and T are inverses of each other. Consider the following definition.

Definition 5.17: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ be linear transformations. Suppose that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x}$$

and

$$(T \circ S)(\vec{x}) = \vec{x}$$

Then, S is called an inverse of T and T is called an inverse of S . Geometrically, they reverse the action of each other.

The following theorem is crucial, as it claims that the above inverse transformations are unique.

Theorem 5.18: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation induced by the matrix A . Then T has an inverse transformation if and only if the matrix A is invertible. In this case, the inverse transformation is unique and denoted $T^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$. T^{-1} is induced by the matrix A^{-1} .

Consider the following example.

Example 5.19: Inverse of a Transformation

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Show that T^{-1} exists and find the matrix B which it is induced by.

Solution. Since the matrix A is invertible, it follows that the transformation T is invertible. Therefore, T^{-1} exists.

You can verify that A^{-1} is given by:

$$A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

Therefore the linear transformation T^{-1} is induced by the matrix A^{-1} . □

5.3.1. EXERCISES

1. Show that if a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then it is **always** the case that $T(\vec{0}) = \vec{0}$.
2. Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and S a linear transformation induced by $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
3. Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is a linear transformation induced by the matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.
4. Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ and S a linear transformation induced by $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
5. Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. Find the matrix of T^{-1} .
6. Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$. Find the matrix of T^{-1} .

7. Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Find the matrix of T^{-1} .

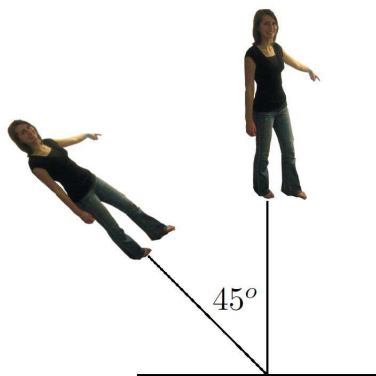
5.4 SPECIAL LINEAR TRANSFORMATIONS IN \mathbb{R}^2

Outcomes

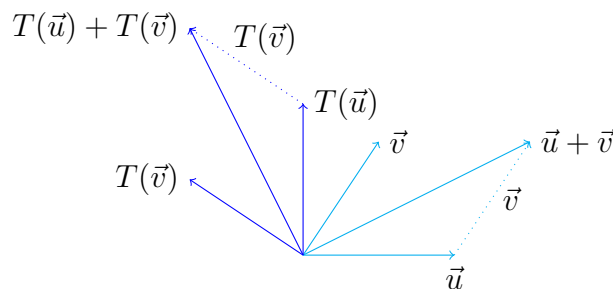
- A. Find the matrix of rotations and reflections in \mathbb{R}^2 and determine the action of each on a vector in \mathbb{R}^2 .

In this section, we will examine some special examples of linear transformations in \mathbb{R}^2 including rotations and reflections. We will use the geometric descriptions of vector addition and scalar multiplication discussed earlier to show that a rotation of vectors through an angle and reflection of a vector across a line are examples of linear transformations.

First, consider the rotation of a vector through an angle. Such a rotation would achieve something like the following if applied to each vector from $(0,0)$ to the point in the picture corresponding to the person shown standing upright.



More generally, denote a transformation given by a rotation by T . Why is such a transformation linear? Consider the following picture which illustrates a rotation. Let \vec{u}, \vec{v} denote vectors.



Let's consider how to obtain $T(\vec{u} + \vec{v})$. Simply, you add $T(\vec{u})$ and $T(\vec{v})$. Here is why. If you add $T(\vec{u})$ to $T(\vec{v})$ you get the diagonal of the parallelogram determined by $T(\vec{u})$ and $T(\vec{v})$, as this action is our usual vector addition. Now, suppose we first add \vec{u} and \vec{v} , and then apply the transformation T to $\vec{u} + \vec{v}$. Hence, we find $T(\vec{u} + \vec{v})$. As shown in the diagram, this will result in the same vector. In other words, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.

This is because the rotation preserves all angles between the vectors as well as their lengths. In particular, it preserves the shape of this parallelogram. Thus both $T(\vec{u}) + T(\vec{v})$ and $T(\vec{u} + \vec{v})$ give the same vector. It follows that T distributes across addition of the vectors of \mathbb{R}^2 .

Similarly, if k is a scalar, it follows that $T(k\vec{u}) = kT(\vec{u})$. Thus rotations are an example of a linear transformation by Definition 5.2.

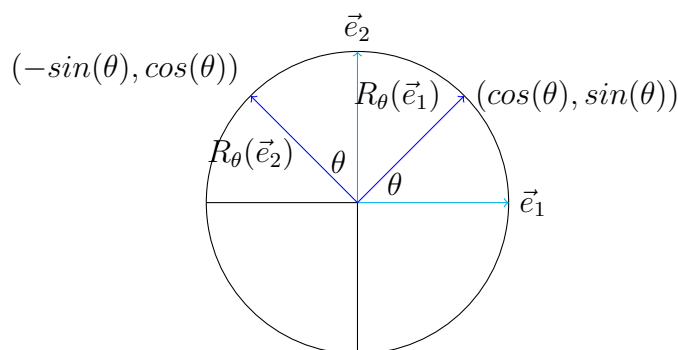
The following theorem gives the matrix of a linear transformation which rotates all vectors through an angle of θ .

Theorem 5.20: Rotation

Let $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation given by rotating vectors through an angle of θ . Then the matrix A of R_θ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Proof. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.



From Theorem 5.5, we need to find $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, and use these as the columns of the matrix A of T . We can use \cos, \sin of the angle θ to find the coordinates of $R_\theta(\vec{e}_1)$ as shown in the above picture. The coordinates of $R_\theta(\vec{e}_2)$ also follow from trigonometry. Thus

$$R_\theta(\vec{e}_1) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, R_\theta(\vec{e}_2) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Therefore, from Theorem 5.5,

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

We can also prove this algebraically without the use of the above picture. The definition of $(\cos(\theta), \sin(\theta))$ is as the coordinates of the point of $R_\theta(\vec{e}_1)$. Now the point of the vector \vec{e}_2 is exactly $\pi/2$ further along the unit circle from the point of \vec{e}_1 , and therefore after rotation through an angle of θ the coordinates x and y of the point of $R_\theta(\vec{e}_2)$ are given by

$$(x, y) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2)) = (-\sin \theta, \cos \theta)$$

□

Consider the following example.

Example 5.21: Rotation in \mathbb{R}^2

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote rotation through $\pi/2$. Find the matrix of $R_{\frac{\pi}{2}}$. Then, find $R_{\frac{\pi}{2}}(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution. By Theorem 5.20, the matrix of $R_{\frac{\pi}{2}}$ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

To find $R_{\frac{\pi}{2}}(\vec{x})$, we multiply the matrix of $R_{\frac{\pi}{2}}$ by \vec{x} as follows

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

□

We now look at an example of a linear transformation involving two angles.

Example 5.22: The Rotation Matrix of the Sum of Two Angles

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ . Hence the linear transformation rotates all vectors through an angle of $\theta + \phi$.

Solution. Let $R_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta + \phi$. Then to obtain $R_{\theta+\phi}$, we first apply R_ϕ and then R_θ where R_ϕ is the linear transformation which rotates through an angle of ϕ and R_θ is the linear transformation which rotates through an angle of θ . Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ , and A_θ , it follows that for every \vec{u}

$$R_{\theta+\phi}(\vec{u}) = A_{\theta+\phi}\vec{u} = A_\theta A_\phi \vec{u} = R_\theta R_\phi(\vec{u})$$

Notice the order of the matrices here!

Consequently, you must have

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} = A_\theta A_\phi \end{aligned}$$

The usual matrix multiplication yields

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & -\cos \theta \sin \phi - \sin \theta \cos \phi \\ \sin \theta \cos \phi + \cos \theta \sin \phi & \cos \theta \cos \phi - \sin \theta \sin \phi \end{bmatrix} \\ &= A_\theta A_\phi \end{aligned}$$

Don't these look familiar? They are the usual trigonometric identities for the sum of two angles derived here using linear algebra concepts.

□

Here we have focused on rotations in two dimensions. However, you can consider rotations and other geometric concepts in any number of dimensions. This is one of the major advantages of linear algebra. You can break down a difficult geometrical procedure into small steps, each corresponding to multiplication by an appropriate matrix. Then by multiplying the matrices, you can obtain a single matrix which can give you numerical information on the results of applying the given sequence of simple procedures.

Linear transformations which reflect vectors across a line are a second important type of transformations in \mathbb{R}^2 . Consider the following theorem.

Theorem 5.23: Reflection

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation given by reflecting vectors over the line $\vec{y} = m\vec{x}$. Then the matrix of Q_m is given by

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Consider the following example.

Example 5.24: Reflection in \mathbb{R}^2

Let $Q_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection over the line $\vec{y} = 2\vec{x}$. Then Q_2 is a linear transformation. Find the matrix of Q_2 . Then, find $Q_2(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution. By Theorem 5.23, the matrix of Q_2 is given by

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(2)^2} \begin{bmatrix} 1-(2)^2 & 2(2) \\ 2(2) & (2)^2-1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}$$

To find $Q_2(\vec{x})$ we multiply \vec{x} by the matrix of Q_2 as follows:

$$\frac{1}{5} \begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{19}{5} \\ \frac{2}{5} \end{bmatrix}$$

□

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 5.25: Rotation Followed by a Reflection

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of $\pi/6$ and then reflecting through the x axis.

Solution. By Theorem 5.20, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Reflecting across the x axis is the same action as reflecting vectors over the line $\vec{y} = m\vec{x}$ with $m = 0$. By Theorem 5.23, the matrix for the transformation which reflects all vectors through the x axis is

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(0)^2} \begin{bmatrix} 1-(0)^2 & 2(0) \\ 2(0) & (0)^2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x axis is given by

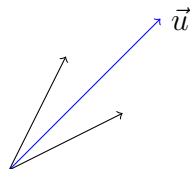
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

□

5.4.1. EXERCISES

1. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/3$.
2. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/4$.
3. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $-\pi/3$.
4. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $2\pi/3$.
5. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/12$. **Hint:** Note that $\pi/12 = \pi/3 - \pi/4$.
6. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $2\pi/3$ and then reflects across the x axis.
7. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/3$ and then reflects across the x axis.
8. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/4$ and then reflects across the x axis.
9. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.
10. Find the matrix for the linear transformation which reflects every vector in \mathbb{R}^2 across the x axis and then rotates every vector through an angle of $\pi/4$.
11. Find the matrix for the linear transformation which reflects every vector in \mathbb{R}^2 across the y axis and then rotates every vector through an angle of $\pi/4$.
12. Find the matrix for the linear transformation which reflects every vector in \mathbb{R}^2 across the x axis and then rotates every vector through an angle of $\pi/6$.
13. Find the matrix for the linear transformation which reflects every vector in \mathbb{R}^2 across the y axis and then rotates every vector through an angle of $\pi/6$.
14. Find the matrix for the linear transformation which rotates every vector in \mathbb{R}^2 through an angle of $5\pi/12$. **Hint:** Note that $5\pi/12 = 2\pi/3 - \pi/4$.
15. Find the matrix of the linear transformation which rotates every vector in \mathbb{R}^3 counter clockwise about the z axis when viewed from the positive z axis through an angle of 30° and then reflects through the xy plane.

16. Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector in \mathbb{R}^2 . Find the matrix which reflects all vectors across this vector, as shown in the following picture.



Hint: Notice that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ for some θ . First rotate through $-\theta$. Next reflect through the x axis. Finally rotate through θ .

5.5 LINEAR TRANSFORMATIONS WHICH ARE ONE TO ONE OR ONTO

Outcomes

A. Determine if a linear transformation is onto or one to one.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. We define the **range** or **image** of T as the set of vectors of \mathbb{R}^m which are of the form $T(\vec{x})$ (equivalently, $A\vec{x}$) for some $\vec{x} \in \mathbb{R}^n$. It is common to write $T\mathbb{R}^n$, $T(\mathbb{R}^n)$, or $\text{Im}(T)$ to denote these vectors.

Lemma 5.26: Range of a Matrix Transformation

Let A be an $m \times n$ matrix where A_1, \dots, A_n denote the columns of A . Then, for a

vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in \mathbb{R}^n ,

$$A\vec{x} = \sum_{k=1}^n x_k A_k$$

Therefore, $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Proof. This follows from the definition of matrix multiplication in Definition 2.13. \square

This section is devoted to studying two important characterizations of linear transformations, called one to one and onto. We define them now.

Definition 5.27: One to One

Suppose \vec{x}_1 and \vec{x}_2 are vectors in \mathbb{R}^n . A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called **one to one** (often written as $1 - 1$) if whenever $\vec{x}_1 \neq \vec{x}_2$ it follows that :

$$T(\vec{x}_1) \neq T(\vec{x}_2)$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is $1 - 1$ if it never takes two different vectors to the same vector.

The second important characterization is called onto.

Definition 5.28: Onto

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called **onto** if whenever $\vec{x}_2 \in \mathbb{R}^m$ there exists $\vec{x}_1 \in \mathbb{R}^n$ such that $T(\vec{x}_1) = \vec{x}_2$.

We often call a linear transformation which is one-to-one an **injection**. Similarly, a linear transformation which is onto is often called a **surjection**.

The following proposition is an important result.

Proposition 5.29: One to One

Let $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation induced by the $m \times n$ matrix A . Then T_A is one to one if and only if $T_A(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Proof. First note that we can rewrite the statement “ $T_A(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$ ” in terms of the matrix A as “ $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$ ”. Therefore we can prove this theorem using A .

We need to prove two things here. First, we will prove that if A is one to one, then $A\vec{x} = \vec{0}$ implies that $\vec{x} = \vec{0}$. Second, we will show that if $A\vec{x} = \vec{0}$ implies that $\vec{x} = \vec{0}$, then it follows that A is one to one.

First note that $A\vec{0} = A(\vec{0} + \vec{0}) = A\vec{0} + A\vec{0}$ and so $A\vec{0} = \vec{0}$.

Now suppose A is one to one and $A\vec{x} = \vec{0}$. We need to show that this implies $\vec{x} = \vec{0}$. Since A is one to one, by Definition 5.27 A can only map one vector to the zero vector $\vec{0}$. Now $A\vec{x} = \vec{0}$ and $A\vec{0} = \vec{0}$, so it follows that $\vec{x} = \vec{0}$. Thus if A is one to one and $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Next assume that $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$. We need to show that A is one to one. Suppose $A\vec{x} = A\vec{y}$. Then $A\vec{x} - A\vec{y} = \vec{0}$. Hence $A\vec{x} - A\vec{y} = A(\vec{x} - \vec{y}) = \vec{0}$. However, we have assumed that $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$. This means that whenever A times a vector equals $\vec{0}$, that vector is also equal to $\vec{0}$. Therefore, $\vec{x} - \vec{y} = \vec{0}$ and so $\vec{x} = \vec{y}$. Thus A is one to one by Definition 5.27. \square

Note that this proposition says that if $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$ then A is one to one if and only if whenever

$$0 = \sum_{k=1}^n c_k A_k$$

it follows that each scalar $c_k = 0$.

We will now take a look at an example of a one to one and onto linear transformation.

Example 5.30: A One to One and Onto Linear Transformation

Suppose

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Then, $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation. Is T onto? Is it one to one?

Solution. Recall that because T can be expressed as matrix multiplication, we know that T is a linear transformation. We will start by looking at onto. So suppose $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Does there exist $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$? If so, then since $\begin{bmatrix} a \\ b \end{bmatrix}$ is an arbitrary vector in \mathbb{R}^2 , it will follow that T is onto.

This question is familiar to you. It is asking whether there is a solution to the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

This is the same thing as asking for a solution to the following system of equations.

$$\begin{aligned} x + y &= a \\ x + 2y &= b \end{aligned}$$

Set up the augmented matrix and row reduce.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2a - b \\ 0 & 1 & b - a \end{array} \right] \quad (5.4)$$

You can see from this point that the system has a solution. Therefore, we have shown that for any a, b , there is a $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Thus T is onto.

Now we want to know if T is one to one. By Lemma 5.29 it is enough to show that $A\vec{x} = 0$ implies $\vec{x} = 0$. Consider the system $A\vec{x} = 0$ given by:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the same as the system given by

$$\begin{aligned} x + y &= 0 \\ x + 2y &= 0 \end{aligned}$$

We need to show that the solution to this system is $x = 0$ and $y = 0$. By setting up the augmented matrix and row reducing, we end up with

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

This tells us that $x = 0$ and $y = 0$. Returning to the original system, this says that if

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words, $A\vec{x} = 0$ implies that $\vec{x} = 0$. By Proposition 5.29, A is one to one, and so T is also one to one.

We also could have seen that T is one to one from our above solution for onto. By looking at the matrix given by 5.4, you can see that there is a **unique** solution given by $x = 2a - b$ and $y = b - a$. Therefore, there is only one vector, specifically $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2a - b \\ b - a \end{bmatrix}$ such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Hence by Definition 5.27, T is one to one. \square

Consider the following important definition.

Definition 5.31: Isomorphism

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called an **isomorphism** if it is **both** one to one and onto.

The above Example 5.30 demonstrated that the given transformation T is both one to one and onto. We can now say that this transformation is an isomorphism.

5.5.1. EXERCISES

1. Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Is T one to one? Is T onto?

2. Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Is T one to one? Is T onto?

3. Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

Is T one to one? Is T onto?

4. Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 0 & 2 \\ 2 & 4 & -6 \end{bmatrix}$$

Is T one to one? Is T onto?

5. Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.
6. Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m . Show that the transformation T determined by A maps \mathbb{R}^n onto \mathbb{R}^m . **Hint:** The vectors $\vec{e}_1, \dots, \vec{e}_m$ occur as columns in the reduced row-echelon form for A .
7. Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n . Show that A is one to one. **Hint:** If not, there exists a vector, \vec{x} such that $A\vec{x} = 0$, and this implies at least one column of A is a linear combination of the others. Show this would require the rank to be less than n .
8. Explain why an $n \times n$ matrix A is both one to one and onto if and only if its rank is n .

5.6 THE GENERAL SOLUTION OF A LINEAR SYSTEM

Outcomes

- A. Use linear transformations to determine the particular solution and general solution to a system of equations.
- B. Find the kernel of a linear transformation.

Recall the definition of a linear transformation discussed above. T is a **linear transformation** if whenever \vec{x}, \vec{y} are vectors and k, p are scalars,

$$T(k\vec{x} + p\vec{y}) = kT(\vec{x}) + pT(\vec{y})$$

Thus linear transformations distribute across addition and pass scalars to the outside.

It turns out that we can use linear transformations to solve linear systems of equations. Indeed given a system of linear equations of the form $A\vec{x} = \vec{b}$, one may rephrase this as $T(\vec{x}) = \vec{b}$ where T is the linear transformation T_A induced by the coefficient matrix A . With this in mind consider the following definition.

Definition 5.32: Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form

$$T(\vec{x}) = \vec{b}$$

If $T(\vec{x}_p) = \vec{b}$, then \vec{x}_p is called a **particular solution** of the linear system.

Recall that a system is called homogeneous if every equation in the system is equal to 0. Suppose we represent a homogeneous system of equations by $T(\vec{x}) = \vec{0}$. It turns out that the \vec{x} for which $T(\vec{x}) = \vec{0}$ are part of a special set called the **null space** of T . We may also refer to the null space as the **kernel** of T , and we write $\ker(T)$.

Consider the following definition.

Definition 5.33: Null Space or Kernel of a Linear Transformation

Let T be a linear transformation. Define

$$\ker(T) = \left\{ \vec{x} : T(\vec{x}) = \vec{0} \right\}$$

The kernel, $\ker(T)$ consists of the set of all vectors \vec{x} for which $T(\vec{x}) = \vec{0}$. This is also called the **null space** of T .

We may also refer to the kernel of T as the **solution space** of the equation $T(\vec{x}) = \vec{0}$. Consider the following example.

Example 5.34: The Kernel of the Derivative

Let $\frac{d}{dx}$ denote the linear transformation defined on f , the functions which are defined on \mathbb{R} and have a continuous derivative. Find $\ker\left(\frac{d}{dx}\right)$.

Solution. The example asks for functions f which the property that $\frac{df}{dx} = 0$. As you may know from calculus, these functions are the constant functions. Thus $\ker\left(\frac{d}{dx}\right)$ is the set of constant functions. \square

Definition 5.33 states that $\ker(T)$ is the set of solutions to the equation,

$$T(\vec{x}) = \vec{0}$$

Since we can write $T(\vec{x})$ as $A\vec{x}$, you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by $A\vec{x} = \vec{b}$, and consider the related homogeneous system. By this, we mean that we replace \vec{b} by $\vec{0}$ and look at $A\vec{x} = \vec{0}$. It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that $T(\vec{x}) = A\vec{x}$.

Theorem 5.35: Particular Solution and General Solution

Suppose \vec{x}_p is a solution to the linear system given by ,

$$T(\vec{x}) = \vec{b}$$

Then if \vec{y} is any other solution to $T(\vec{x}) = \vec{b}$, there exists $\vec{x}_0 \in \ker(T)$ such that

$$\vec{y} = \vec{x}_p + \vec{x}_0$$

Hence, every solution to the linear system can be written as a sum of a particular solution, \vec{x}_p , and a solution \vec{x}_0 to the associated homogeneous system given by $T(\vec{x}) = \vec{0}$.

Proof. Consider $\vec{y} - \vec{x}_p = \vec{y} + (-1)\vec{x}_p$. Then $T(\vec{y} - \vec{x}_p) = T(\vec{y}) - T(\vec{x}_p)$. Since \vec{y} and \vec{x}_p are both solutions to the system, it follows that $T(\vec{y}) = \vec{b}$ and $T(\vec{x}_p) = \vec{b}$.

Hence, $T(\vec{y}) - T(\vec{x}_p) = \vec{b} - \vec{b} = \vec{0}$. Let $\vec{x}_0 = \vec{y} - \vec{x}_p$. Then, $T(\vec{x}_0) = \vec{0}$ so \vec{x}_0 is a solution to the associated homogeneous system and so is in $\ker(T)$. \square

Sometimes people remember the above theorem in the following form. The solutions to the system $T(\vec{x}) = \vec{b}$ are given by $\vec{x}_p + \ker(T)$ where \vec{x}_p is a particular solution to $T(\vec{x}) = \vec{b}$.

For now, we have been speaking about the kernel or null space of a linear transformation T . However, we know that every linear transformation T is determined by some matrix A . Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example 5.36: The Null Space of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

Find $\ker(A)$. Equivalently, find the solutions to the system of equations $A\vec{x} = \vec{0}$.

Solution. We are asked to find $\{\vec{x} : A\vec{x} = \vec{0}\}$. In other words we want to solve the system,

$A\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then this amounts to solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the linear system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + y + z + 2w &= 0 \\ 4x + 5y + 7z + 2w &= 0 \end{aligned}$$

To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 4 & 5 & 7 & 2 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This yields $x = \frac{1}{3}z - \frac{4}{3}w$ and $y = \frac{2}{3}w - \frac{5}{3}z$. Since $\ker(A)$ consists of the solutions to this system, it consists vectors of the form,

$$\begin{bmatrix} \frac{1}{3}z - \frac{4}{3}w \\ \frac{2}{3}w - \frac{5}{3}z \\ z \\ w \end{bmatrix} = z \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

□

Consider the following example.

Example 5.37: A General Solution

The **general solution** of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 25 \end{bmatrix}$$

given that $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ is one solution.

Solution. Note the matrix of this system is the same as the matrix in Example 5.36. Therefore, from Theorem 5.35, you will obtain all solutions to the above linear system by adding a particular solution \vec{x}_p to the solutions of the associated homogeneous system, \vec{x} . One particular solution is given above by

$$\vec{x}_p = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad (5.5)$$

Using this particular solution along with the solutions found in Example 5.36, we obtain the following solutions,

$$z \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, any solution to the above linear system is of this form. \square

5.6.1. EXERCISES

1. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2. Using Problem 1 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

3. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Using Problem 3 find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

5. Write the solution set of the following system as a linear combination of vectors.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6. Using Problem 5 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

7. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

8. Using Problem 7 find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

9. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

10. Using Problem 9 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

11. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

12. Using Problem 11 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \end{bmatrix}$$

13. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

14. Using Problem 13 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

15. Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

16. Using Problem 15 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

17. Suppose $A\vec{x} = \vec{b}$ has a solution. Explain why the solution is unique precisely when $A\vec{x} = \vec{0}$ has only the trivial solution.

6. COMPLEX NUMBERS

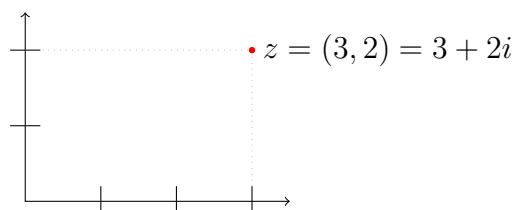
6.1 COMPLEX NUMBERS

Outcomes

- A. Understand the geometric significance of a complex number as a point in the plane.
- B. Prove algebraic properties of addition and multiplication of complex numbers, and apply these properties. Understand the action of taking the conjugate of a complex number.
- C. Understand the absolute value of a complex number and how to find it as well as its geometric significance.

Although very powerful, the real numbers are inadequate to solve equations such as $x^2 + 1 = 0$, and this is where complex numbers come in. We define the number i as the imaginary number such that $i^2 = -1$, and define complex numbers as those of the form $z = a + bi$ where a and b are real numbers. We call this the standard form, or Cartesian form, of the complex number z . Then, we refer to a as the *real* part of z , and b as the *imaginary* part of z . It turns out that such numbers not only solve the above equation, but in fact also solve any polynomial of degree at least 1 with complex coefficients. This property, called the Fundamental Theorem of Algebra, is sometimes referred to by saying \mathbb{C} is algebraically closed. Gauss is usually credited with giving a proof of this theorem in 1797 but many others worked on it and the first completely correct proof was due to Argand in 1806.

Just as a real number can be considered as a point on the line, a complex number $z = a + bi$ can be considered as a point (a, b) in the plane whose x coordinate is a and whose y coordinate is b . For example, in the following picture, the point $z = 3 + 2i$ can be represented as the point in the plane with coordinates $(3, 2)$.



Addition of complex numbers is defined as follows.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

This addition obeys all the usual properties as the following theorem indicates.

Theorem 6.1: Properties of Addition of Complex Numbers

Let z, w , and v be complex numbers. Then the following properties hold.

- *Commutative Law for Addition*

$$z + w = w + z$$

- *Additive Identity*

$$z + 0 = z$$

- *Existence of Additive Inverse*

For each $z \in \mathbb{C}$, there exists $-z \in \mathbb{C}$ such that $z + (-z) = 0$
In fact if $z = a + bi$, then $-z = -a - bi$.

- *Associative Law for Addition*

$$(z + w) + v = z + (w + v)$$

The proof of this theorem is left as an exercise for the reader.

Now, multiplication of complex numbers is defined the way you would expect, recalling that $i^2 = -1$.

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + i^2bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Consider the following examples.

Example 6.2: Multiplication of Complex Numbers

- $(2 - 3i)(-3 + 4i) = 6 + 17i$
- $(4 - 7i)(6 - 2i) = 10 - 50i$
- $(-3 + 6i)(5 - i) = -9 + 33i$

The following are important properties of multiplication of complex numbers.

Theorem 6.3: Properties of Multiplication of Complex Numbers

Let z, w and v be complex numbers. Then, the following properties of multiplication hold.

- Commutative Law for Multiplication

$$zw = wz$$

- Associative Law for Multiplication

$$(zw)v = z(wv)$$

- Multiplicative Identity

$$1z = z$$

- Existence of Multiplicative Inverse

For each $z \neq 0$, there exists z^{-1} such that $zz^{-1} = 1$

- Distributive Law

$$z(w + v) = zw + zv$$

You may wish to verify some of these statements. The real numbers also satisfy the above axioms, and in general any mathematical structure which satisfies these axioms is called a field. There are many other fields, in particular even finite ones particularly useful for cryptography, and the reason for specifying these axioms is that linear algebra is all about fields and we can do just about anything in this subject using any field. Although here, the fields of most interest will be the familiar field of real numbers, denoted as \mathbb{R} , and the field of complex numbers, denoted as \mathbb{C} .

An important construction regarding complex numbers is the complex conjugate denoted by a horizontal line above the number, \bar{z} . It is defined as follows.

Definition 6.4: Conjugate of a Complex Number

Let $z = a + bi$ be a complex number. Then the **conjugate** of z , written \bar{z} is given by

$$\overline{a + bi} = a - bi$$

Geometrically, the action of the conjugate is to reflect a given complex number across the x axis. Algebraically, it changes the sign on the imaginary part of the complex number. Therefore, for a real number a , $\bar{a} = a$.

Example 6.5: Conjugate of a Complex Number

- If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$, i.e., $\overline{3 + 4i} = 3 - 4i$.
- $\overline{-2 + 5i} = -2 - 5i$.
- $\bar{i} = -i$.
- $\bar{7} = 7$.

Consider the following computation.

$$\begin{aligned}(\overline{a + bi})(a + bi) &= (a - bi)(a + bi) \\ &= a^2 + b^2 - (ab - ab)i = a^2 + b^2\end{aligned}$$

Notice that there is no imaginary part in the product, thus multiplying a complex number by its conjugate results in a real number.

Theorem 6.6: Properties of the Conjugate

Let z and w be complex numbers. Then, the following properties of the conjugate hold.

- $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
- $\overline{(zw)} = \bar{z} \bar{w}$.
- $\overline{(\bar{z})} = z$.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.
- z is real if and only if $\bar{z} = z$.

Division of complex numbers is defined as follows. Let $z = a + bi$ and $w = c + di$ be complex numbers such that c, d are not both zero. Then the quotient z divided by w is

$$\begin{aligned}\frac{z}{w} = \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.\end{aligned}$$

In other words, the quotient $\frac{z}{w}$ is obtained by multiplying both top and bottom of $\frac{z}{w}$ by \bar{w} and then simplifying the expression.

Example 6.7: Division of Complex Numbers

•

$$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-i^2} = -i$$

•

$$\frac{2-i}{3+4i} = \frac{2-i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{(6-4) + (-3-8)i}{3^2 + 4^2} = \frac{2-11i}{25} = \frac{2}{25} - \frac{11}{25}i$$

•

$$\frac{1-2i}{-2+5i} = \frac{1-2i}{-2+5i} \times \frac{-2-5i}{-2-5i} = \frac{(-2-10) + (4-5)i}{2^2 + 5^2} = -\frac{12}{29} - \frac{1}{29}i$$

Interestingly every nonzero complex number $a + bi$ has a unique multiplicative inverse. In other words, for a nonzero complex number z , there exists a number z^{-1} (or $\frac{1}{z}$) so that $zz^{-1} = 1$. Note that $z = a + bi$ is nonzero exactly when $a^2 + b^2 \neq 0$, and its inverse can be written in standard form as defined now.

Definition 6.8: Inverse of a Complex Number

Let $z = a + bi$ be a complex number. Then the multiplicative inverse of z , written z^{-1} exists if and only if $a^2 + b^2 \neq 0$ and is given by

$$z^{-1} = \frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Note that we may write z^{-1} as $\frac{1}{z}$. Both notations represent the multiplicative inverse of the complex number z . Consider now an example.

Example 6.9: Inverse of a Complex Number

Consider the complex number $z = 2 + 6i$. Then z^{-1} is defined, and

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2+6i} \\ &= \frac{1}{2+6i} \times \frac{2-6i}{2-6i} \\ &= \frac{2-6i}{2^2+6^2} \\ &= \frac{2-6i}{40} \\ &= \frac{1}{20} - \frac{3}{20}i\end{aligned}$$

You can always check your answer by computing zz^{-1} .

Another important construction of complex numbers is that of the absolute value, also called the modulus. Consider the following definition.

Definition 6.10: Absolute Value

The absolute value, or modulus, of a complex number, denoted $|z|$ is defined as follows.

$$|a + bi| = \sqrt{a^2 + b^2}$$

Thus, if z is the complex number $z = a + bi$, it follows that

$$|z| = (z\bar{z})^{1/2}$$

Also from the definition, if $z = a + bi$ and $w = c + di$ are two complex numbers, then $|zw| = |z| |w|$. Take a moment to verify this.

The triangle inequality is an important property of the absolute value of complex numbers. There are two useful versions which we present here, although the first one is officially called the triangle inequality.

Proposition 6.11: Triangle Inequality

Let z, w be complex numbers.

The following two inequalities hold for any complex numbers z, w :

$$\begin{aligned} |z + w| &\leq |z| + |w| \\ ||z| - |w|| &\leq |z - w| \end{aligned}$$

The first one is called the Triangle Inequality.

Proof. Let $z = a + bi$ and $w = c + di$. First note that

$$z\bar{w} = (a + bi)(c - di) = ac + bd + (bc - ad)i$$

and so $|ac + bd| \leq |z\bar{w}| = |z| |w|$.

Then,

$$\begin{aligned} |z + w|^2 &= (a + c + i(b + d))(a + c - i(b + d)) \\ &= (a + c)^2 + (b + d)^2 = a^2 + c^2 + 2ac + b^2 + d^2 + 2bd \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

Taking the square root, we have that

$$|z + w| \leq |z| + |w|$$

so this verifies the triangle inequality.

To get the second inequality, write

$$z = z - w + w, \quad w = w - z + z$$

and so by the first form of the inequality we get both:

$$|z| \leq |z - w| + |w|, \quad |w| \leq |z - w| + |z|$$

Hence, both $|z| - |w|$ and $|w| - |z|$ are no larger than $|z - w|$. This proves the second version because $||z| - |w||$ is one of $|z| - |w|$ or $|w| - |z|$. \square

With this definition, it is important to note the following. You may wish to take the time to verify this remark.

Let $z = a + bi$ and $w = c + di$. Then $|z - w| = \sqrt{(a - c)^2 + (b - d)^2}$. Thus the distance between the point in the plane determined by the ordered pair (a, b) and the ordered pair (c, d) equals $|z - w|$ where z and w are as just described.

For example, consider the distance between $(2, 5)$ and $(1, 8)$. Letting $z = 2 + 5i$ and $w = 1 + 8i$, $z - w = 1 - 3i$, $(z - w)(\overline{z - w}) = (1 - 3i)(1 + 3i) = 10$ so $|z - w| = \sqrt{10}$.

Recall that we refer to $z = a + bi$ as the standard form of the complex number. In the next section, we examine another form in which we can express the complex number.

6.1.1. EXERCISES

1. Let $z = 2 + 7i$ and let $w = 3 - 8i$. Compute the following.

- (a) $z + w$
- (b) $z - 2w$
- (c) zw
- (d) $\frac{z}{w}$

2. Let $z = 1 - 4i$. Compute the following.

- (a) \overline{z}
- (b) z^{-1}
- (c) $|z|$

3. Let $z = 3 + 5i$ and $w = 2 - i$. Compute the following.

- (a) \overline{zw}
- (b) $|zw|$
- (c) $z^{-1}w$

4. If z is a complex number, show there exists a complex number w with $|w| = 1$ and $wz = |z|$.

5. If z, w are complex numbers prove $\overline{zw} = \overline{z} \overline{w}$ and then show by induction that $\overline{z_1 \cdots z_m} = \overline{z_1} \cdots \overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.
6. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where all the a_k are real numbers. Suppose also that $p(z) = 0$ for some $z \in \mathbb{C}$. Show it follows that $p(\overline{z}) = 0$ also.
7. I claim that $1 = -1$. Here is why.

$$-1 = i^2 = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$$

This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?

6.2 POLAR FORM

Outcomes

- A. Convert a complex number from standard form to polar form, and from polar form to standard form.

In the previous section, we identified a complex number $z = a + bi$ with a point (a, b) in the coordinate plane. There is another form in which we can express the same number, called the *polar form*. The polar form is the focus of this section. It will turn out to be very useful if not crucial for certain calculations as we shall soon see.

Suppose $z = a + bi$ is a complex number, and let $r = \sqrt{a^2 + b^2} = |z|$. Recall that r is the **modulus** of z . Note first that

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2 + b^2}{r^2} = 1$$

and so $\left(\frac{a}{r}, \frac{b}{r}\right)$ is a point on the unit circle. Therefore, there exists an angle θ (in radians) such that

$$\cos \theta = \frac{a}{r}, \quad \sin \theta = \frac{b}{r}$$

In other words θ is an angle such that $a = r \cos \theta$ and $b = r \sin \theta$, that is $\theta = \cos^{-1}(a/r)$ and $\theta = \sin^{-1}(b/r)$. We call this angle θ the **argument** of z .

We often speak of the **principal argument** of z . This is the unique angle $\theta \in (-\pi, \pi]$ such that

$$\cos \theta = \frac{a}{r}, \quad \sin \theta = \frac{b}{r}$$

The polar form of the complex number $z = a + bi = r(\cos \theta + i \sin \theta)$ is for convenience written as:

$$z = re^{i\theta}$$

where θ is the argument of z .

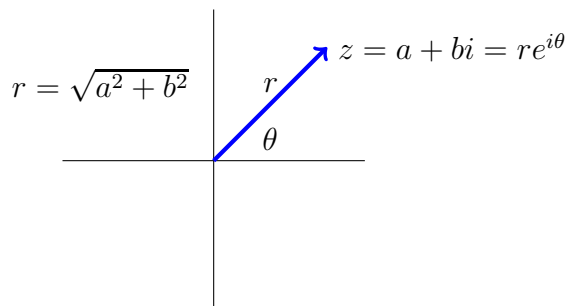
Definition 6.12: Polar Form of a Complex Number

Let $z = a + bi$ be a complex number. Then the **polar form** of z is written as

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and θ is the argument of z .

When given $z = re^{i\theta}$, the identity $e^{i\theta} = \cos \theta + i \sin \theta$ will convert z back to standard form. Here we think of $e^{i\theta}$ as a short cut for $\cos \theta + i \sin \theta$. This is all we will need in this course, but in reality $e^{i\theta}$ can be considered as the complex equivalent of the exponential function where this turns out to be a true equality.



Thus we can convert any complex number in the standard (Cartesian) form $z = a + bi$ into its polar form. Consider the following example.

Example 6.13: Standard to Polar Form

Let $z = 2 + 2i$ be a complex number. Write z in the polar form

$$z = re^{i\theta}$$

Solution. First, find r . By the above discussion, $r = \sqrt{a^2 + b^2} = |z|$. Therefore,

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Now, to find θ , we plot the point $(2, 2)$ and find the angle from the positive x axis to the line between this point and the origin. In this case, $\theta = 45^\circ = \frac{\pi}{4}$. That is we found the unique angle θ such that $\theta = \cos^{-1}(1/\sqrt{2})$ and $\theta = \sin^{-1}(1/\sqrt{2})$.

Note that in polar form, we always express angles in radians, not degrees.

Hence, we can write z as

$$z = 2\sqrt{2}e^{i\frac{\pi}{4}}$$

□

Notice that the standard and polar forms are completely equivalent. That is not only can we transform a complex number from standard form to its polar form, we can also take a complex number in polar form and convert it back to standard form.

Example 6.14: Polar to Standard Form

Let $z = 2e^{2\pi i/3}$. Write z in the standard form

$$z = a + bi$$

Solution. Let $z = 2e^{2\pi i/3}$ be the polar form of a complex number. Recall that $e^{i\theta} = \cos \theta + i \sin \theta$. Therefore using standard values of sin and cos we get:

$$\begin{aligned} z = 2e^{i2\pi/3} &= 2(\cos(2\pi/3) + i \sin(2\pi/3)) \\ &= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -1 + \sqrt{3}i \end{aligned}$$

which is the standard form of this complex number. □

You can always verify your answer by converting it back to polar form and ensuring you reach the original answer.

6.2.1. EXERCISES

1. Let $z = 3 + 3i$ be a complex number written in standard form. Convert z to polar form, and write it in the form $z = re^{i\theta}$.
2. Let $z = 2i$ be a complex number written in standard form. Convert z to polar form, and write it in the form $z = re^{i\theta}$.
3. Let $z = 4e^{\frac{2\pi}{3}i}$ be a complex number written in polar form. Convert z to standard form, and write it in the form $z = a + bi$.
4. Let $z = -1e^{\frac{\pi}{6}i}$ be a complex number written in polar form. Convert z to standard form, and write it in the form $z = a + bi$.
5. If z and w are two complex numbers and the polar form of z involves the angle θ while the polar form of w involves the angle ϕ , show that in the polar form for zw the angle involved is $\theta + \phi$.

6.3 ROOTS OF COMPLEX NUMBERS

Outcomes

- A. Understand De Moivre's theorem and be able to use it to find the roots of a complex number.

A fundamental identity is the formula of De Moivre with which we begin this section.

Theorem 6.15: De Moivre's Theorem

For any positive integer n , we have

$$(e^{i\theta})^n = e^{in\theta}$$

Thus for any real number $r > 0$ and any positive integer n , we have:

$$(r(\cos \theta + i \sin \theta))^n = r^n (\cos n\theta + i \sin n\theta)$$

Proof. The proof is by induction on n . It is clear the formula holds if $n = 1$. Suppose it is true for n . Then, consider $n + 1$.

$$(r(\cos \theta + i \sin \theta))^{n+1} = (r(\cos \theta + i \sin \theta))^n (r(\cos \theta + i \sin \theta))$$

which by induction equals

$$\begin{aligned} &= r^{n+1} (\cos n\theta + i \sin n\theta) (\cos \theta + i \sin \theta) \\ &= r^{n+1} ((\cos n\theta \cos \theta - \sin n\theta \sin \theta) + i (\sin n\theta \cos \theta + \cos n\theta \sin \theta)) \\ &= r^{n+1} (\cos (n+1)\theta + i \sin (n+1)\theta) \end{aligned}$$

by the formulas for the cosine and sine of the sum of two angles. \square

The process used in the previous proof, called *mathematical induction* is very powerful in Mathematics and Computer Science and explored in more detail in the Appendix.

Now, consider a corollary of Theorem 6.15.

Corollary 6.16: Roots of Complex Numbers

Let z be a non zero complex number. Then there are always exactly k many k^{th} roots of z in \mathbb{C} .

Proof. Let $z = a + bi$ and let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of the complex number. By De Moivre's theorem, a complex number

$$w = re^{i\alpha} = r(\cos \alpha + i \sin \alpha)$$

is a k^{th} root of z if and only if

$$w^k = (re^{i\alpha})^k = r^k e^{ik\alpha} = r^k (\cos k\alpha + i \sin k\alpha) = |z| (\cos \theta + i \sin \theta)$$

This requires $r^k = |z|$ and so $r = |z|^{1/k}$. Also, both $\cos(k\alpha) = \cos \theta$ and $\sin(k\alpha) = \sin \theta$. This can only happen if

$$k\alpha = \theta + 2\ell\pi$$

for ℓ an integer. Thus

$$\alpha = \frac{\theta + 2\ell\pi}{k}, \ell = 0, 1, 2, \dots, k-1$$

and so the k^{th} roots of z are of the form

$$|z|^{1/k} \left(\cos \left(\frac{\theta + 2\ell\pi}{k} \right) + i \sin \left(\frac{\theta + 2\ell\pi}{k} \right) \right), \ell = 0, 1, 2, \dots, k-1$$

Since the cosine and sine are periodic of period 2π , there are exactly k distinct numbers which result from this formula. \square

The procedure for finding the k k^{th} roots of $z \in \mathbb{C}$ is as follows.

Procedure 6.17: Finding Roots of a Complex Number

Let w be a complex number. We wish to find the n^{th} roots of w , that is all z such that $z^n = w$.

There are n distinct n^{th} roots and they can be found as follows:

1. Express both z and w in polar form $z = re^{i\theta}$, $w = se^{i\phi}$. Then $z^n = w$ becomes:

$$(re^{i\theta})^n = r^n e^{in\theta} = se^{i\phi}$$

We need to solve for r and θ .

2. Solve the following two equations:

$$r^n = s$$

$$e^{in\theta} = e^{i\phi} \tag{6.1}$$

3. The solutions to $r^n = s$ are given by $r = \sqrt[n]{s}$.

4. The solutions to $e^{in\theta} = e^{i\phi}$ are given by:

$$n\theta = \phi + 2\pi\ell, \text{ for } \ell = 0, 1, 2, \dots, n-1$$

or

$$\theta = \frac{\phi}{n} + \frac{2}{n}\pi\ell, \text{ for } \ell = 0, 1, 2, \dots, n-1$$

5. Using the solutions r, θ to the equations given in (6.1) construct the n^{th} roots of the form $z = re^{i\theta}$.

Notice that once the roots are obtained in the final step, they can then be converted to standard form if necessary. Let's consider an example of this concept. Note that according to Corollary 6.16, there are exactly 3 cube roots of a complex number.

Example 6.18: Finding Cube Roots

Find the three cube roots of i . In other words find all z such that $z^3 = i$.

Solution. First, convert each number to polar form: $z = re^{i\theta}$ and $i = 1e^{i\pi/2}$. The equation now becomes

$$(re^{i\theta})^3 = r^3 e^{3i\theta} = 1e^{i\pi/2}$$

Therefore, the two equations that we need to solve are $r^3 = 1$ and $3i\theta = i\pi/2$. Given that $r \in \mathbb{R}$ and $r^3 = 1$ it follows that $r = 1$.

Solving the second equation is as follows. First divide by i . Then, since the argument of

i is not unique we write $3\theta = \pi/2 + 2\pi\ell$ for $\ell = 0, 1, 2$.

$$3\theta = \pi/2 + 2\pi\ell \text{ for } \ell = 0, 1, 2$$

$$\theta = \pi/6 + \frac{2}{3}\pi\ell \text{ for } \ell = 0, 1, 2$$

For $\ell = 0$:

$$\theta = \pi/6 + \frac{2}{3}\pi(0) = \pi/6$$

For $\ell = 1$:

$$\theta = \pi/6 + \frac{2}{3}\pi(1) = \frac{5}{6}\pi$$

For $\ell = 2$:

$$\theta = \pi/6 + \frac{2}{3}\pi(2) = \frac{3}{2}\pi$$

Therefore, the three roots are given by

$$1e^{i\pi/6}, 1e^{i\frac{5}{6}\pi}, 1e^{i\frac{3}{2}\pi}$$

Written in standard form, these roots are, respectively,

$$\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} + i\frac{1}{2}, -i$$

□

The ability to find k^{th} roots can also be used to factor some polynomials.

Example 6.19: Solving a Polynomial Equation

Factor the polynomial $x^3 - 27$.

Solution. First find the cube roots of 27. By the above procedure, these cube roots are 3 , $3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$. You may wish to verify this using the above steps.

Therefore, $x^3 - 27 =$

$$(x - 3) \left(x - 3 \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2} \right) \right) \left(x - 3 \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2} \right) \right)$$

Note also $\left(x - 3 \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2} \right) \right) \left(x - 3 \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2} \right) \right) = x^2 + 3x + 9$ and so

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$

where the quadratic polynomial $x^2 + 3x + 9$ cannot be factored without using complex numbers. □

Note that even though the polynomial $x^3 - 27$ has all real coefficients, it has some complex zeros, $3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$. These zeros are complex conjugates of each other. It is always the case that if a polynomial has real coefficients and a complex root, it will also have a root equal to the complex conjugate.

6.3.1. EXERCISES

1. Give the complete solution to $x^4 + 16 = 0$.
2. Find the complex cube roots of 8.
3. Find the four fourth roots of 16.
4. De Moivre's theorem says $[r(\cos t + i \sin t)]^n = r^n(\cos nt + i \sin nt)$ for n a positive integer. Does this formula continue to hold for all integers n , even negative integers? Explain.
5. Factor $x^3 + 8$ as a product of linear factors. **Hint:** Use the result of 2.
6. Write $x^3 + 27$ in the form $(x + 3)(x^2 + ax + b)$ where $x^2 + ax + b$ cannot be factored any more using only real numbers.
7. Completely factor $x^4 + 16$ as a product of linear factors. **Hint:** Use the result of 3.
8. Factor $x^4 + 16$ as the product of two quadratic polynomials each of which cannot be factored further without using complex numbers.
9. If n is an integer, is it always true that $(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta)$? Explain.
10. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial and it has n zeros,

$$z_1, z_2, \dots, z_n$$

listed according to multiplicity. (z is a root of multiplicity m if the polynomial $f(x) = (x - z)^m$ divides $p(x)$ but $(x - z)f(x)$ does not.) Show that

$$p(x) = a_n (x - z_1)(x - z_2) \cdots (x - z_n)$$

6.4 THE QUADRATIC FORMULA

Outcomes

A. Use the Quadratic Formula to find the complex roots of a quadratic equation.

The roots (or solutions) of a quadratic equation $ax^2 + bx + c = 0$ where a, b, c are real numbers are obtained by solving the familiar quadratic formula given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When working with real numbers, we cannot solve this formula if $b^2 - 4ac < 0$. However, complex numbers allow us to find square roots of negative numbers, and the quadratic formula remains valid for finding roots of the corresponding quadratic equation. In this case there are exactly two distinct (complex) square roots of $b^2 - 4ac$, which are $i\sqrt{4ac - b^2}$ and $-i\sqrt{4ac - b^2}$.

Here is an example.

Example 6.20: Solutions to Quadratic Equation

Find the solutions to $x^2 + 2x + 5 = 0$.

Solution. In terms of the quadratic equation above, $a = 1$, $b = 2$, and $c = 5$. Therefore, we can use the quadratic formula with these values, which becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

Solving this equation, we see that the solutions are given by

$$x = \frac{-2i \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

We can verify that these are solutions of the original equation. We will show $x = -1 + 2i$ and leave $x = -1 - 2i$ as an exercise.

$$\begin{aligned} x^2 + 2x + 5 &= (-1 + 2i)^2 + 2(-1 + 2i) + 5 \\ &= 1 - 4i - 4 - 2 + 4i + 5 \\ &= 0 \end{aligned}$$

Hence $x = -1 + 2i$ is a solution. □

What if the coefficients of the quadratic equation are actually complex numbers? Does the formula hold even in this case? The answer is yes. This is a hint on how to do Problem 4 below, a special case of the fundamental theorem of algebra, and an ingredient in the proof of some versions of this theorem.

Consider the following example.

Example 6.21: Solutions to Quadratic Equation

Find the solutions to $x^2 - 2ix - 5 = 0$.

Solution. In terms of the quadratic equation above, $a = 1$, $b = -2i$, and $c = -5$. Therefore, we can use the quadratic formula with these values, which becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2i \pm \sqrt{(-2i)^2 - 4(1)(-5)}}{2(1)}$$

Solving this equation, we see that the solutions are given by

$$x = \frac{2i \pm \sqrt{-4 + 20}}{2} = \frac{2i \pm 4}{2} = i \pm 2$$

We can verify that these are solutions of the original equation. We will show $x = i + 2$ and leave $x = i - 2$ as an exercise.

$$\begin{aligned}x^2 - 2ix - 5 &= (i + 2)^2 - 2i(i + 2) - 5 \\&= -1 + 4i + 4 + 2 - 4i - 5 \\&= 0\end{aligned}$$

Hence $x = i + 2$ is a solution. □

We conclude this section by stating an essential theorem.

Theorem 6.22: The Fundamental Theorem of Algebra

Any polynomial of degree at least 1 with complex coefficients has a root which is a complex number.

6.4.1. EXERCISES

1. Show that $1 + i, 2 + i$ are the only two roots to

$$p(x) = x^2 - (3 + 2i)x + (1 + 3i)$$

Hence complex zeros do not necessarily come in conjugate pairs if the coefficients of the equation are not real.

2. Give the solutions to the following quadratic equations having real coefficients.

(a) $x^2 - 2x + 2 = 0$

(b) $3x^2 + x + 3 = 0$

(c) $x^2 - 6x + 13 = 0$

(d) $x^2 + 4x + 9 = 0$

(e) $4x^2 + 4x + 5 = 0$

3. Give the solutions to the following quadratic equations having complex coefficients.

(a) $x^2 + 2x + 1 + i = 0$

(b) $4x^2 + 4ix - 5 = 0$

(c) $4x^2 + (4 + 4i)x + 1 + 2i = 0$

(d) $x^2 - 4ix - 5 = 0$

(e) $3x^2 + (1 - i)x + 3i = 0$

4. Prove the fundamental theorem of algebra for quadratic polynomials having coefficients in \mathbb{C} . That is, show that an equation of the form $ax^2 + bx + c = 0$ where a, b, c are complex numbers, $a \neq 0$ has a complex solution. **Hint:** Consider the fact, noted earlier that the expressions given from the quadratic formula do in fact serve as solutions.

7. SPECTRAL THEORY

7.1 EIGENVALUES AND EIGENVECTORS OF A MATRIX

Outcomes

- A. Describe eigenvalues geometrically and algebraically.
- B. Find eigenvalues and eigenvectors for a square matrix.

Spectral Theory refers to the study of eigenvalues and eigenvectors of a matrix. It is of fundamental importance in many areas and is the subject of our study for this chapter.

7.1.1. DEFINITION OF EIGENVECTORS AND EIGENVALUES

In this section, we will work with the entire set of complex numbers, denoted by \mathbb{C} . Recall that the real numbers, \mathbb{R} are contained in the complex numbers, so the discussions in this section apply to both real and complex numbers.

To illustrate the idea behind what will be discussed, consider the following example.

Example 7.1: Eigenvectors and Eigenvalues

Let

$$A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix}$$

Compute the product AX for

$$X = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

What do you notice about AX in each of these products?

Solution. First, compute AX for

$$X = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

This product is given by

$$AX = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

In this case, the product AX resulted in a vector which is equal to 10 times the vector X . In other words, $AX = 10X$.

Let's see what happens in the next product. Compute AX for the vector

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This product is given by

$$AX = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In this case, the product AX resulted in a vector equal to 0 times the vector X , $AX = 0X$. Perhaps this matrix is such that AX results in kX , for every vector X . However, consider

$$\begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 38 \\ -11 \end{bmatrix}$$

In this case, AX did not result in a vector of the form kX for some scalar k . □

There is something special about the first two products calculated in Example 7.1. Notice that for each, $AX = kX$ where k is some scalar. When this equation holds for some X and k , we call the scalar k an **eigenvalue** of A . We often use the special symbol λ instead of k when referring to eigenvalues. In Example 7.1, the values 10 and 0 are eigenvalues for the matrix A and we can label these as $\lambda_1 = 10$ and $\lambda_2 = 0$.

When $AX = \lambda X$ for some $X \neq 0$, we call such an X an **eigenvector** of the matrix A . The eigenvectors of A are associated to an eigenvalue. Hence, if λ_1 is an eigenvalue of A and $AX = \lambda_1 X$, we can label this eigenvector as X_1 . Note again that in order to be an eigenvector, X must be nonzero.

There is also a geometric significance to eigenvectors. When you have a **nonzero** vector which, when multiplied by a matrix results in another vector which is parallel to the first or equal to $\mathbf{0}$, this vector is called an eigenvector of the matrix. This is the meaning when the vectors are in \mathbb{R}^n .

The formal definition of eigenvalues and eigenvectors is as follows.

Definition 7.2: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a **nonzero vector** for which

$$AX = \lambda X \quad (7.1)$$

for some scalar λ . Then λ is called an **eigenvalue** of the matrix A and X is called an **eigenvector** of A associated with λ , or a λ -eigenvector of A .

The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the **spectrum** of A .

The eigenvectors of a matrix A are those vectors X for which multiplication by A results in a vector in the same direction or opposite direction to X . Since the zero vector 0 has no direction this would make no sense for the zero vector. As noted above, 0 is never allowed to be an eigenvector.

Let's look at eigenvectors in more detail. Suppose X satisfies 7.1. Then

$$\begin{aligned} AX - \lambda X &= 0 \\ \text{or} \\ (A - \lambda I)X &= 0 \end{aligned}$$

for some $X \neq 0$. Equivalently you could write $(\lambda I - A)X = 0$, which is more commonly used. Hence, when we are looking for eigenvectors, we are looking for nontrivial solutions to this homogeneous system of equations!

Recall that the solutions to a homogeneous system of equations consist of basic solutions, and the linear combinations of those basic solutions. In this context, we call the basic solutions of the equation $(\lambda I - A)X = 0$ **basic eigenvectors**. It follows that any (nonzero) linear combination of basic eigenvectors is again an eigenvector.

Suppose the matrix $(\lambda I - A)$ is invertible, so that $(\lambda I - A)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned} X &= IX \\ &= ((\lambda I - A)^{-1}(\lambda I - A))X \\ &= (\lambda I - A)^{-1}((\lambda I - A)X) \\ &= (\lambda I - A)^{-1}0 \\ &= 0 \end{aligned}$$

This claims that $X = 0$. However, we have required that $X \neq 0$. Therefore $(\lambda I - A)$ cannot have an inverse!

Recall from Theorem 3.33 that if a matrix is not invertible, then its determinant is equal to 0. Therefore we can conclude that

$$\det(\lambda I - A) = 0 \quad (7.2)$$

Note that this is equivalent to $\det(A - \lambda I) = 0$.

The expression $\det(xI - A)$ is a polynomial (in the variable x) called the **characteristic polynomial of A** , and $\det(xI - A) = 0$ is called the **characteristic equation**. For this reason we may also refer to the eigenvalues of A as **characteristic values**, but the former is often used for historical reasons.

The following theorem claims that the roots of the characteristic polynomial are the eigenvalues of A . Thus when 7.2 holds, A has a nonzero eigenvector.

Theorem 7.3: The Existence of an Eigenvector

Let A be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of A and thus there exists a nonzero vector $X \in \mathbb{C}^n$ such that $AX = \lambda X$.

Proof. For A an $n \times n$ matrix, the method of Laplace Expansion demonstrates that $\det(\lambda I - A)$ is a polynomial of degree n . As such, the equation 7.2 has a solution $\lambda \in \mathbb{C}$ by the Fundamental Theorem of Algebra. The fact that λ is an eigenvalue follows from Theorem 3.33 and is left as an exercise. \square

7.1.2. FINDING EIGENVECTORS AND EIGENVALUES

Now that eigenvalues and eigenvectors have been defined, we will study how to find them for a matrix A .

First, consider the following definition.

Definition 7.4: Multiplicity of an Eigenvalue

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(xI - A)$. Then, the multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

For example, suppose the characteristic polynomial of A is given by $(x - 2)^2$. Solving for the roots of this polynomial, we set $(x - 2)^2 = 0$ and solve for x . We find that $\lambda = 2$ is a root that occurs twice. Hence, in this case, $\lambda = 2$ is an eigenvalue of A of multiplicity equal to 2.

We will now look at how to find the eigenvalues and eigenvectors for a matrix A in detail. The steps used are summarized in the following procedure.

Procedure 7.5: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(xI - A) = 0$.
2. For each λ , find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X .

We will explore these steps further in the following example.

Example 7.6: Find the Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$. Find its eigenvalues and eigenvectors.

Solution. We will use Procedure 7.5. First we find the eigenvalues of A by solving the equation

$$\det(xI - A) = 0$$

This gives

$$\begin{aligned} \det\left(x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right) &= 0 \\ \det \begin{bmatrix} x+5 & -2 \\ 7 & x-4 \end{bmatrix} &= 0 \end{aligned}$$

Computing the determinant as usual, the result is

$$x^2 + x - 6 = 0$$

Solving this equation, we find that $\lambda_1 = 2$ and $\lambda_2 = -3$.

Now we need to find the basic eigenvectors for each λ . First we will find the eigenvectors for $\lambda_1 = 2$. We wish to find all vectors $X \neq 0$ such that $AX = 2X$. These are the solutions to $(2I - A)X = 0$.

$$\begin{aligned} \left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The solution is any vector of the form

$$\begin{bmatrix} \frac{2}{7}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$$

Multiplying this vector by 7 we obtain a simpler description for the solution to this system, given by

$$t \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_1 = 2$ as

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

To check, we verify that $AX = 2X$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

Next we will repeat this process to find the basic eigenvector for $\lambda_2 = -3$. We wish to find all vectors $X \neq 0$ such that $AX = -3X$. These are the solutions to $((-3)I - A)X = 0$.

$$\begin{aligned} \left((-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 7 & -7 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The solution is any vector of the form

$$\begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_2 = -3$ as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To check, we verify that $AX = -3X$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct. □

The following is an example using Procedure 7.5 for a 3×3 matrix.

Example 7.7: Find the Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$$

Solution. We will use Procedure 7.5. First we need to find the eigenvalues of A . Recall that they are the solutions of the equation

$$\det(xI - A) = 0$$

In this case the equation is

$$\det \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) = 0$$

which becomes

$$\det \begin{bmatrix} x-5 & 10 & 5 \\ -2 & x-14 & -2 \\ 4 & 8 & x-6 \end{bmatrix} = 0$$

Using Laplace Expansion, compute this determinant and simplify. The result is the following equation.

$$(x-5)(x^2-20x+100) = 0$$

Solving this equation, we find that the eigenvalues are $\lambda_1 = 5, \lambda_2 = 10$ and $\lambda_3 = 10$. Notice that 10 is a root of multiplicity two due to

$$x^2 - 20x + 100 = (x-10)^2$$

Therefore, $\lambda_2 = 10$ is an eigenvalue of multiplicity two.

Now that we have found the eigenvalues for A , we can compute the eigenvectors.

First we will find the basic eigenvectors for $\lambda_1 = 5$. In other words, we want to find all non-zero vectors X so that $AX = 5X$. This requires that we solve the equation $(5I - A)X = 0$ for X as follows.

$$\left(5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you need to find the solution to

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By now this is a familiar problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\left[\begin{array}{ccc|c} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so the solution is any vector of the form

$$\begin{bmatrix} \frac{5}{4}s \\ -\frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

where $s \in \mathbb{R}$. If we multiply this vector by 4, we obtain a simpler description for the solution to this system, as given by

$$t \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \tag{7.3}$$

where $t \in \mathbb{R}$. Here, the basic eigenvector is given by

$$X_1 = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

Notice that we cannot let $t = 0$ here, because this would result in the zero vector and eigenvectors are never equal to 0! Other than this value, every other choice of t in 7.3 results in an eigenvector.

It is a good idea to check your work! To do so, we will take the original matrix and multiply by the basic eigenvector X_1 . We check to see if we get $5X_1$.

$$\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

This is what we wanted, so we know that our calculations were correct.

Next we will find the basic eigenvectors for $\lambda_2, \lambda_3 = 10$. These vectors are the basic solutions to the equation,

$$\left(10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you must find the solutions to

$$\begin{bmatrix} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{array} \right]$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so the eigenvectors are of the form

$$\begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Note that you can't pick t and s both equal to zero because this would result in the zero vector and eigenvectors are never equal to zero.

Here, there are two basic eigenvectors, given by

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Taking any (nonzero) linear combination of X_2 and X_3 will also result in an eigenvector for the eigenvalue $\lambda = 10$. As in the case for $\lambda = 5$, always check your work! For the first basic eigenvector, we can check $AX_2 = 10X_2$ as follows.

$$\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This is what we wanted. Checking the second basic eigenvector, X_3 , is left as an exercise. \square

It is important to remember that for any eigenvector X , $X \neq 0$. However, it is possible to have eigenvalues equal to zero. This is illustrated in the following example.

Example 7.8: A Zero Eigenvalue

Let

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A .

Solution. First we find the eigenvalues of A . We will do so using Definition 7.2.

In order to find the eigenvalues of A , we solve the following equation.

$$\det(xI - A) = \det \begin{bmatrix} x-2 & -2 & 2 \\ -1 & x-3 & 1 \\ 1 & -1 & x-1 \end{bmatrix} = 0$$

This reduces to $x^3 - 6x^2 + 8x = 0$. You can verify that the solutions are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$. Notice that while eigenvectors can never equal 0, it is possible to have an eigenvalue equal to 0.

Now we will find the basic eigenvectors. For $\lambda_1 = 0$, we need to solve the equation $(0I - A)X = 0$. This equation becomes $-AX = 0$, and so the augmented matrix for finding the solutions is given by

$$\left[\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are of the form $t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where $t \neq 0$ and the basic eigenvector is given by

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We can verify that this eigenvector is correct by checking that the equation $AX_1 = 0X_1$ holds. The product AX_1 is given by

$$AX_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This clearly equals $0X_1$, so the equation holds. Hence, $AX_1 = 0X_1$ and so 0 is an eigenvalue of A .

Computing the other basic eigenvectors is left as an exercise. \square

In the following sections, we examine ways to simplify this process of finding eigenvalues and eigenvectors by using properties of special types of matrices.

7.1.3. EIGENVALUES AND EIGENVECTORS FOR SPECIAL TYPES OF MATRICES

There are three special kinds of matrices which we can use to simplify the process of finding eigenvalues and eigenvectors. Throughout this section, we will discuss similar matrices, elementary matrices, as well as triangular matrices.

We begin with a definition.

Definition 7.9: Similar Matrices

Let A and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

$$A = P^{-1}BP$$

Then A and B are called **similar matrices**.

It turns out that we can use the concept of similar matrices to help us find the eigenvalues of matrices. Consider the following lemma.

Lemma 7.10: Similar Matrices and Eigenvalues

Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A, B have the same eigenvalues.

Proof. We need to show two things. First, we need to show that if $A = P^{-1}BP$, then A and B have the same eigenvalues. Secondly, we show that if A and B have the same eigenvalues, then $A = P^{-1}BP$.

Here is the proof of the first statement. Suppose $A = P^{-1}BP$ and λ is an eigenvalue of A , that is $AX = \lambda X$ for some $X \neq 0$. Then

$$P^{-1}BPX = \lambda X$$

and so

$$BPX = \lambda PX$$

Since P is one to one and $X \neq 0$, it follows that $PX \neq 0$. Here, PX plays the role of the eigenvector in this equation. Thus λ is also an eigenvalue of B . One can similarly verify that any eigenvalue of B is also an eigenvalue of A , and thus both matrices have the same eigenvalues as desired.

Proving the second statement is similar and is left as an exercise. \square

Note that this proof also demonstrates that the eigenvectors of A and B will (generally) be *different*. We see in the proof that $AX = \lambda X$, while $B(PX) = \lambda(PX)$. Therefore, for an eigenvalue λ , A will have the eigenvector X while B will have the eigenvector PX .

The second special type of matrices we discuss in this section is elementary matrices. Recall from Definition 2.43 that an elementary matrix E is obtained by applying one row operation to the identity matrix.

It is possible to use elementary matrices to simplify a matrix before searching for its eigenvalues and eigenvectors. This is illustrated in the following example.

Example 7.11: Simplify Using Elementary Matrices

Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix}$$

Solution. This matrix has big numbers and therefore we would like to simplify as much as possible before computing the eigenvalues.

We will do so using row operations. First, add 2 times the second row to the third row. To do so, left multiply A by $E(2, 2)$. Then right multiply A by the inverse of $E(2, 2)$ as illustrated.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

By Lemma 7.10, the resulting matrix has the same eigenvalues as A where here, the matrix $E(2, 2)$ plays the role of P .

We do this step again, as follows. In this step, we use the elementary matrix obtained by adding -3 times the second row to the first row.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} \quad (7.4)$$

Again by Lemma 7.10, this resulting matrix has the same eigenvalues as A . At this point, we can easily find the eigenvalues. Let

$$B = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, we find the eigenvalues of B (and therefore of A) by solving the equation $\det(xI - B) = 0$. You should verify that this equation becomes

$$(x + 2)(x + 2)(x - 3) = 0$$

Solving this equation results in eigenvalues of $\lambda_1 = -2$, $\lambda_2 = -2$, and $\lambda_3 = 3$. Therefore, these are also the eigenvalues of A .

□

Through using elementary matrices, we were able to create a matrix for which finding the eigenvalues was easier than for A . At this point, you could go back to the original matrix A and solve $(\lambda I - A)X = 0$ to obtain the eigenvectors of A .

Notice that when you multiply on the right by an elementary matrix, you are doing the column operation defined by the elementary matrix. In 7.4 multiplication by the elementary

matrix on the right merely involves taking three times the first column and adding to the second. Thus, without referring to the elementary matrices, the transition to the new matrix in 7.4 can be illustrated by

$$\begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -9 & 15 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

The third special type of matrix we will consider in this section is the triangular matrix. Recall Definition 3.12 which states that an upper (lower) triangular matrix contains all zeros below (above) the main diagonal. Remember that finding the determinant of a triangular matrix is a simple procedure of taking the product of the entries on the main diagonal.. It turns out that there is also a simple way to find the eigenvalues of a triangular matrix.

In the next example we will demonstrate that the eigenvalues of a triangular matrix are the entries on the main diagonal.

Example 7.12: Eigenvalues for a Triangular Matrix

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$. Find the eigenvalues of A .

Solution. We need to solve the equation $\det(xI - A) = 0$ as follows

$$\det(xI - A) = \det \begin{bmatrix} x-1 & -2 & -4 \\ 0 & x-4 & -7 \\ 0 & 0 & x-6 \end{bmatrix} = (x-1)(x-4)(x-6) = 0$$

Solving the equation $(x-1)(x-4)(x-6) = 0$ for x results in the eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of the original matrix. \square

The same result is true for lower triangular matrices. For any triangular matrix, the eigenvalues are equal to the entries on the main diagonal. To find the eigenvectors of a triangular matrix, we use the usual procedure.

In the next section, we explore an important process involving the eigenvalues and eigenvectors of a matrix.

7.1.4. EXERCISES

1. If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1} . More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m .
2. If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA .

- Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose X is an eigenvector of B . Show that then AX must also be an eigenvector for B .
- Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.
- Show that if $AX = \lambda X$ and $AY = \lambda Y$, then whenever k, p are scalars,

$$A(kX + pY) = \lambda(kX + pY)$$

Does this imply that $kX + pY$ is an eigenvector? Explain.

- Suppose A is a 3×3 matrix and the following information is available.

$$\begin{aligned} A \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} &= 0 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} &= -2 \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \end{aligned}$$

Find $A \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$.

- Suppose A is a 3×3 matrix and the following information is available.

$$\begin{aligned} A \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} &= 1 \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \\ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix} &= 2 \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix} \end{aligned}$$

Find $A \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}$.

8. Suppose A is a 3×3 matrix and the following information is available.

$$A \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix}$$

Find $A \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$.

9. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -6 & -92 & 12 \\ 0 & 0 & 0 \\ -2 & -31 & 4 \end{bmatrix}$$

One eigenvalue is -2 .

10. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & -17 & -6 \\ 0 & 0 & 0 \\ 1 & 9 & 3 \end{bmatrix}$$

One eigenvalue is 1 .

11. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

One eigenvalue is -3 .

12. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

One eigenvalue is -2 .

13. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ -8 & -11 & -4 \\ 10 & 11 & 3 \end{bmatrix}$$

One eigenvalue is -3.

14. If A is the matrix of a linear transformation which rotates all vectors in \mathbb{R}^2 through 60° , explain why A cannot have any real eigenvalues. Is there an angle such that rotation through this angle would have a real eigenvalue? What eigenvalues would be obtainable in this way?
15. Let A be the 2×2 matrix of the linear transformation which rotates all vectors in \mathbb{R}^2 through an angle of θ . For which values of θ does A have a real eigenvalue?
16. Is it possible for a nonzero matrix to have only 0 as an eigenvalue?
17. Let A be the 2×2 matrix of the linear transformation which rotates all vectors in \mathbb{R}^2 through an angle of θ . For which values of θ does A have a real eigenvalue?
18. Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.
19. Let T be the linear transformation which rotates all vectors in \mathbb{R}^2 counterclockwise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.
20. Let T be the linear transformation which reflects all vectors in \mathbb{R}^3 through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

7.2 DIAGONALIZATION

Outcomes

- A. Determine when it is possible to diagonalize a matrix.
- B. When possible, diagonalize a matrix.

We begin this section by recalling Definition 7.9 of similar matrices. Recall that if A, B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

The following are important properties of similar matrices.

Proposition 7.13: Properties of Similarity

Define \sim for $n \times n$ matrices A, B and C by $A \sim B$ if A is similar to B . Then

- $A \sim A$
- If $A \sim B$ then $B \sim A$
- If $A \sim B$ and $B \sim C$ then $A \sim C$

Proof. It is clear that $A \sim A$, taking $P = I$.

Now, if $A \sim B$, then for some P invertible,

$$A = P^{-1}BP$$

and so

$$PAP^{-1} = B$$

But then

$$(P^{-1})^{-1}AP^{-1} = B$$

which shows that $B \sim A$ by Definition 7.9.

Now suppose $A \sim B$ and $B \sim C$. Then there exist invertible matrices P, Q such that

$$A = P^{-1}BP, \quad B = Q^{-1}CQ$$

Then,

$$A = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$$

showing that A is similar to C by Definition 7.9. \square

When a matrix is similar to a diagonal matrix, the matrix is said to be diagonalizable. We define a diagonal matrix D as a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D , then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

where $*$ is a number which might not be zero.

The following is the formal definition of a diagonalizable matrix.

Definition 7.14: Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be **diagonalizable** if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Notice that the above equation can be rearranged as $A = PDP^{-1}$. Suppose we wanted to compute A^{100} . By diagonalizing A first it suffices to then compute $(PDP^{-1})^{100}$, which reduces to $PD^{100}P^{-1}$. This last computation is much simpler than A^{100} . While this process is described in detail later, it provides motivation for diagonalization.

7.2.1. DIAGONALIZING A MATRIX

The most important theorem about diagonalizability is the following major result.

Theorem 7.15: Eigenvectors and Diagonalizable Matrices

An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where the X_k are eigenvectors of A .

Moreover if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D .

Proof. Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A . Then P^{-1} is of the form

$$P^{-1} = \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix}$$

where $W_k^T X_j = \delta_{kj}$, which is the Kronecker's symbol discussed earlier.

Then

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix} \begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix} \\ &= \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix} \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{aligned}$$

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where the columns are the X_k and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then

$$AP = PD = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$\begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix} = \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix}$$

showing the X_k are eigenvectors of A and the λ_k are eigenvalues.

□

We demonstrate this concept in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible so must consist of a wide variety of eigenvectors. We achieve this by using basic eigenvectors for the columns of P .

Example 7.16: Diagonalize a Matrix

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution. By Theorem 7.15 we use the eigenvectors of A as the columns of P , and the corresponding eigenvalues of A as the diagonal entries of D .

First, we will find the eigenvalues of A . To do so, we solve $\det(xI - A) = 0$ as follows.

$$\det \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \right) = 0$$

This computation is left as an exercise, and you should verify that the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 6$.

Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda_1, \lambda_2 = 2$. Solving $(2I - A)X = 0$ to find the eigenvectors, we find that the eigenvectors are

$$t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where t, s are scalars. Hence there are two basic eigenvectors which are given by

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

You can verify that the basic eigenvector for $\lambda_3 = 6$ is $X_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Then, we construct the matrix P as follows.

$$P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

That is, the columns of P are the basic eigenvectors of A . Then, you can verify that

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A . We expected this based on Theorem 7.15. Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P . \square

It is possible that a matrix A cannot be diagonalized. In other words, we cannot find an invertible matrix P so that $P^{-1}AP = D$.

Consider the following example.

Example 7.17: A Matrix which cannot be Diagonalized

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and diagonal matrix D so that $P^{-1}AP = D$.

Solution. Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. To find the eigenvectors, we solve the equation $(\lambda I - A)X = 0$. The matrix $(\lambda I - A)$ is given by

$$\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

Substituting in $\lambda = 1$, we have the matrix

$$\begin{bmatrix} 1 - 1 & -1 \\ 0 & 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, solving the equation $(\lambda I - A)X = 0$ involves carrying the following augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Then the eigenvectors are of the form

$$t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the basic eigenvector is

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, the matrix A has one eigenvalue of multiplicity two, but only one basic eigenvector. In order to diagonalize A , we need to construct an invertible 2×2 matrix P . However, because A only has one basic eigenvector, we cannot construct this P . Notice that if we were to use X_1 as both columns of P , P would not be invertible. For this reason, we cannot repeat eigenvectors in P .

Hence this matrix cannot be diagonalized. \square

Recall Definition 7.4 of the multiplicity of an eigenvalue. It turns out that we can determine when a matrix is diagonalizable based on the multiplicity of its eigenvalues. In order for A to be diagonalizable, the number of basic eigenvectors associated with an eigenvalue must be the same number as the multiplicity of the eigenvalue. In Example 7.17, A had one eigenvalue $\lambda = 1$ of multiplicity 2. However, there was only one basic eigenvector associated with this eigenvalue. Therefore, we can see that A is not diagonalizable.

We summarize this in the following theorem.

Theorem 7.18: Diagonalizability Condition

An $n \times n$ matrix A is diagonalizable exactly when the number of basic eigenvectors associated with an eigenvalue is the same number as the multiplicity of that eigenvalue.

You may wonder if there is a need to find P^{-1} , since we can use Theorem 7.15 to construct P and D . We will see this is needed to compute high powers of matrices, which is one of the major applications of diagonalizability.

Before we do so, we first discuss complex eigenvalues.

7.2.2. COMPLEX EIGENVALUES

In some applications, a matrix may have eigenvalues which are complex numbers. For example, this often occurs in differential equations. These questions are approached in the same way as above.

Consider the following example.

Example 7.19: A Real Matrix with Complex Eigenvalues

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A .

Solution. We will first find the eigenvalues as usual by solving the following equation.

$$\det \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \right) = 0$$

This reduces to $(x - 1)(x^2 - 4x + 5) = 0$. The solutions are $\lambda_1 = 1$, $\lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$.

There is nothing new about finding the eigenvectors for $\lambda_1 = 1$ so this is left as an exercise.

Consider now the eigenvalue $\lambda_2 = 2 + i$. As usual, we solve the equation $(\lambda I - A)X = 0$ as given by

$$\left((2 + i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \right) X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, we need to solve the system represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1+i & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{array} \right]$$

We now use our row operations to solve the system. Divide the first row by $(1 + i)$ and then take $-i$ times the second row and add to the third row. This yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Now multiply the second row by $-i$ to obtain the reduced row-echelon form, given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are of the form

$$t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

and the basic eigenvector is given by

$$X_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

As an exercise, verify that the eigenvectors for $\lambda_3 = 2 - i$ are of the form

$$t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Hence, the basic eigenvector is given by

$$X_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

As usual, be sure to check your answers! To verify, we check that $AX_3 = (2 - i)X_3$ as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 - 2i \\ 2 - i \end{bmatrix} = (2 - i) \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Therefore, we know that this eigenvector and eigenvalue are correct. \square

Notice that in Example 7.19, two of the eigenvalues were given by $\lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$. You may recall that these two complex numbers are **conjugates**. It turns out that whenever a matrix containing real entries has a complex eigenvalue λ , it also has an eigenvalue equal to $\bar{\lambda}$, the conjugate of λ .

7.2.3. EXERCISES

1. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

2. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

One eigenvalue is 3. Diagonalize if possible.

3. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

One eigenvalue is -3 . Diagonalize if possible.

4. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 90 & 0 \\ 0 & -2 & 0 \\ 3 & 89 & -2 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

5. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 11 & 45 & 30 \\ 10 & 26 & 20 \\ -20 & -60 & -44 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

6. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 95 & 25 & 24 \\ -196 & -53 & -48 \\ -164 & -42 & -43 \end{bmatrix}$$

One eigenvalue is 5. Diagonalize if possible.

7. Suppose A is an $n \times n$ matrix and let V be an eigenvector such that $AV = \lambda V$. Also suppose the characteristic polynomial of A is

$$\det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)V = 0$$

If A is diagonalizable, give a proof of the Cayley Hamilton theorem based on this. This theorem says A satisfies its characteristic equation,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0$$

8. Suppose the characteristic polynomial of an $n \times n$ matrix A is $1 - x^n$. Find A^{mn} where m is an integer.

9. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -24 & 7 \\ -6 & 5 & -1 \\ -58 & 76 & -20 \end{bmatrix}$$

One eigenvalue is -2 . Diagonalize if possible. **Hint:** This one has some complex eigenvalues.

10. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -25 & 6 \\ -13 & 23 & -4 \\ -91 & 155 & -30 \end{bmatrix}$$

One eigenvalue is 2 . Diagonalize if possible. **Hint:** This one has some complex eigenvalues.

11. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -11 & -12 & 4 \\ 8 & 17 & -4 \\ -4 & 28 & -3 \end{bmatrix}$$

One eigenvalue is 1 . Diagonalize if possible. **Hint:** This one has some complex eigenvalues.

12. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 14 & -12 & 5 \\ -6 & 2 & -1 \\ -69 & 51 & -21 \end{bmatrix}$$

One eigenvalue is -3 . Diagonalize if possible. **Hint:** This one has some complex eigenvalues.

13. Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a + ib$ is a complex eigenvalue having the eigenvector, $X + iY$. Here X and Y are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $X - iY$. **Hint:** You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a + ib$ is a complex number whose conjugate equals $a - ib$.

7.3 APPLICATIONS OF SPECTRAL THEORY

Outcomes

- A. Use diagonalization to find a high power of a matrix.
- B. Use diagonalization to solve dynamical systems.

7.3.1. RAISING A MATRIX TO A HIGH POWER

Suppose we have a matrix A and we want to find A^{50} . One could try to multiply A with itself 50 times, but this is computationally extremely intensive (try it!). However diagonalization allows us to compute high powers of a matrix relatively easily. Suppose A is diagonalizable, so that $P^{-1}AP = D$. We can rearrange this equation to write $A = PDP^{-1}$.

Now, consider A^2 . Since $A = PDP^{-1}$, it follows that

$$A^2 = (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

Similarly,

$$A^3 = (PDP^{-1})^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$$

In general,

$$A^n = (PDP^{-1})^n = PD^nP^{-1}$$

Therefore, we have reduced the problem to finding D^n . In order to compute D^n , then because D is diagonal we only need to raise every entry on the main diagonal of D to the power of n .

Through this method, we can compute large powers of matrices. Consider the following example.

Example 7.20: Raising a Matrix to a High Power

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Find A^{50} .

Solution. We will first diagonalize A . The steps are left as an exercise and you may wish to verify that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$.

The basic eigenvectors corresponding to $\lambda_1, \lambda_2 = 1$ are

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The basic eigenvector corresponding to $\lambda_3 = 2$ is

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now we construct P by using the basic eigenvectors of A as the columns of P . Thus

$$P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then also

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

which you may wish to verify.

Then,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= D \end{aligned}$$

Now it follows by rearranging the equation that

$$A = PDP^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} A^{50} &= PD^{50}P^{-1} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

By our discussion above, D^{50} is found as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} = \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix}$$

It follows that

$$\begin{aligned}
 A^{50} &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 2^{50} & -1 + 2^{50} & 0 \\ 0 & 1 & 0 \\ 1 - 2^{50} & 1 - 2^{50} & 1 \end{bmatrix}
 \end{aligned}$$

□

Through diagonalization, we can efficiently compute a high power of A . Without this, we would be forced to multiply this by hand!

The next section explores another interesting application of diagonalization.

7.3.2. RAISING A SYMMETRIC MATRIX TO A HIGH POWER

We already have seen how to use matrix diagonalization to compute powers of matrices. This requires computing eigenvalues of the matrix A , and finding an invertible matrix of eigenvectors P such that $P^{-1}AP$ is diagonal. In this section we will see that if the matrix A is symmetric (see Definition 2.29), then we can actually find such a matrix P that is an orthogonal matrix of eigenvectors. Thus P^{-1} is simply its transpose P^T , and P^TAP is diagonal. When this happens we say that A is **orthogonally diagonalizable**.

In fact this happens if and only if A is a symmetric matrix as shown in the following important theorem.

Theorem 7.21: Principal Axis Theorem

The following conditions are equivalent for an $n \times n$ matrix A :

1. A is symmetric.
2. A has an orthonormal set of eigenvectors.
3. A is orthogonally diagonalizable.

Proof. The complete proof is beyond this course, but to give an idea assume that A has an orthonormal set of eigenvectors, and let P consist of these eigenvectors as columns. Then $P^{-1} = P^T$, and $P^TAP = D$ a diagonal matrix. But then $A = PDP^T$, and

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

so A is symmetric.

Now given a symmetric matrix A , one shows that eigenvectors corresponding to different eigenvalues are always orthogonal. So it suffices to apply the Gram-Schmidt process on the set of basic eigenvectors of each eigenvalue to obtain an orthonormal set of eigenvectors. □

We demonstrate this in the following example.

Example 7.22: Orthogonal Diagonalization of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix P such that $P^T A P$ is a diagonal matrix.

Solution.

In this case, verify that the eigenvalues are 2 and 1. First we will find an eigenvector for the eigenvalue 2. This involves row reducing the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Finally to obtain an eigenvector of length one (unit eigenvector) we simply divide this vector by its length to yield:

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Next consider the case of the eigenvalue 1. To obtain basic eigenvectors, the matrix which needs to be row reduced in this case is

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Note that all these vectors are automatically orthogonal to eigenvectors corresponding to the first eigenvalue. This follows from the fact that A is symmetric, as mentioned earlier.

We obtain basic eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since they are themselves orthogonal (by luck here) we do not need to use the Gram-Schmidt process and instead simply normalize these vectors to obtain

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

An orthogonal matrix P to orthogonally diagonalize A is then obtained by letting these basic vectors be the columns.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

We verify this works. $P^T A P$ is of the form

$$\begin{aligned} & \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

which is the desired diagonal matrix. □

We can now apply this technique to efficiently compute high powers of a symmetric matrix.

Example 7.23: Powers of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Compute A^7 .

Solution. We found in Example 7.22 that $P^T A P = D$ is diagonal, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus $A = P D P^T$ and $A^7 = P D P^T P D P^T \dots P D P^T = P D^7 P^T$ which gives:

$$\begin{aligned} A^7 &= \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^7 \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^7 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 \\ 0 & \frac{2^7}{2}\sqrt{2} & \frac{2^7}{2}\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2^7+1}{2} & \frac{2^7-1}{2} \\ 0 & \frac{2^7-1}{2} & \frac{2^7+1}{2} \end{bmatrix} \end{aligned}$$

□

7.3.3. MARKOV MATRICES

There are applications which are of great importance which feature a special type of matrix. Matrices in which the columns are non-negative numbers which sum to one are called **Markov matrices**. An important application of Markov matrices is in population migration, as illustrated in the following definition.

Definition 7.24: Migration Matrices

Let n locations be denoted by the numbers $1, 2, \dots, n$. Suppose it is the case that each year the proportion of residents in location j which move to location i is a_{ij} . Also suppose no one escapes or emigrates from without these n locations. This last assumption requires $\sum_i a_{ij} = 1$, and means that the matrix A , such that $A = [a_{ij}]$, is a Markov matrix. In this context, A is also called a **migration matrix**.

Consider the following example which demonstrates this situation.

Example 7.25: Migration Matrix

Let A be a Markov matrix given by

$$A = \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}$$

Verify that A is a Markov matrix and describe the entries of A in terms of population migration.

Solution. The columns of A are comprised of non-negative numbers which sum to 1. Hence, A is a Markov matrix.

Now, consider the entries a_{ij} of A in terms of population. The entry $a_{11} = .4$ is the proportion of residents in location one which stay in location one in a given time period. Entry $a_{21} = .6$ is the proportion of residents in location 1 which move to location 2 in the same time period. Entry $a_{12} = .2$ is the proportion of residents in location 2 which move to location 1. Finally, entry $a_{22} = .8$ is the proportion of residents in location 2 which stay in location 2 in this time period.

Considered as a Markov matrix, these numbers are usually identified with probabilities. Hence, we can say that the probability that a resident of location one will stay in location one in the time period is .4. \square

Observe that in Example 7.25 if there was initially say 15 thousand people in location 1 and 10 thousands in location 2, then after one year there would be $.4 \times 15 + .2 \times 10 = 8$ thousands people in location 1 the following year, and similarly there would be $.6 \times 15 + .8 \times 10 = 17$ thousands people in location 2 the following year.

More generally let $X_n = [x_{1n} \cdots x_{mn}]^T$ where x_{in} is the population of location i at time period n . We call X_n the **state vector at period n** . In particular, we call X_0 the initial state vector. Letting A be the migration matrix, we compute the population in each location i one time period later by AX_n . In order to find the population of location i after k years, we compute the i^{th} component of $A^k X$. This discussion is summarized in the following theorem.

Theorem 7.26: State Vector

Let A be the migration matrix of a population and let X_n be the vector whose entries give the population of each location at time period n . Then X_n is the state vector at period n and it follows that

$$X_{n+1} = AX_n$$

The sum of the entries of X_n will equal the sum of the entries of the initial vector X_0 . Since the columns of A sum to 1, this sum is preserved for every multiplication by A as demonstrated below.

$$\sum_i \sum_j a_{ij} x_j = \sum_j x_j \left(\sum_i a_{ij} \right) = \sum_j x_j$$

Consider the following example.

Example 7.27: Using a Migration Matrix

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations 1, 2, and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 3. Find the population in the three locations after 1, 2, and 10 units of time.

Solution. Using Theorem 7.26 we can find the population in each location using the equation $X_{n+1} = AX_n$. For the population after 1 unit, we calculate $X_1 = AX_0$ as follows.

$$\begin{aligned} X_1 &= AX_0 \\ \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \end{aligned}$$

Therefore after one time period, location 1 has 100 residents, location 2 has 180, and location 3 has 420. Notice that the **total** population is unchanged, it simply migrates within the given locations. We find the locations after two time periods in the same way.

$$\begin{aligned} X_2 &= AX_1 \\ \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \\ &= \begin{bmatrix} 102 \\ 164 \\ 434 \end{bmatrix} \end{aligned}$$

We could progress in this manner to find the populations after 10 time periods. However from our above discussion, we can simply calculate $(A^n X_0)_i$, where n denotes the number of time periods which have passed. Therefore, we compute the populations in each location after 10 units of time as follows.

$$\begin{aligned} X_{10} &= A^{10} X_0 \\ \begin{bmatrix} x_{110} \\ x_{210} \\ x_{310} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 115.085\,829\,22 \\ 120.130\,672\,44 \\ 464.783\,498\,34 \end{bmatrix} \end{aligned}$$

Since we are speaking about populations, we would need to round these numbers to provide a logical answer. Therefore, we can say that after 10 units of time, there will be 115 residents in location one, 120 in location two, and 465 in location three.

□

Suppose we wish to know how many residents will be in a certain location after a very long time. It turns out that if some power of the migration matrix has all positive entries, then there is a vector X_s such that $A^n X_0$ approaches X_s as n becomes very large. Hence as more time passes and n increases, $A^n X_0$ will become closer to the vector X_s .

Consider Theorem 7.26. Let n increase so that X_n approaches X_s . As X_n becomes closer to X_s , so too does X_{n+1} . For sufficiently large n , the statement $X_{n+1} = AX_n$ can be written as $X_s = AX_s$.

This discussion motivates the following theorem.

Theorem 7.28: Steady State Vector

Let A be a migration matrix. Then there exists a **steady state vector** written X_s such that

$$X_s = AX_s$$

where X_s has positive entries which have the same sum as the entries of X_0 . As n increases, the state vectors X_n will approach X_s .

Note that the condition in Theorem 7.28 can be written as $(I - A)X_s = 0$, representing a homogeneous system of equations.

Consider the following example. Notice that it is the same example as the Example 7.27 but here it will involve a longer time frame.

Example 7.29: Populations over the Long Run

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations 1, 2, and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 3. Find the population in the three locations after a long time.

Solution. By Theorem 7.28 the steady state vector X_s can be found by solving the system $(I - A)X_s = 0$.

Thus we need to find a solution to

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 0.4 & 0 & -0.1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -0.2 & -0.2 & 0.1 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.25 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are

$$t \begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix}$$

The initial vector X_0 is given by

$$\begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix}$$

Now all that remains is to choose the value of t such that

$$0.25t + 0.25t + t = 100 + 200 + 400$$

Solving this equation for t yields $t = \frac{1400}{3}$. Therefore the population in the long run is given by

$$\frac{1400}{3} \begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} 116.666 \text{ } 666 \text{ } 666 \text{ } 666 \text{ } 7 \\ 116.666 \text{ } 666 \text{ } 666 \text{ } 666 \text{ } 7 \\ 466.666 \text{ } 666 \text{ } 666 \text{ } 666 \text{ } 7 \end{bmatrix}$$

Again, because we are working with populations, these values need to be rounded. The steady state vector X_s is given by

$$\begin{bmatrix} 117 \\ 117 \\ 466 \end{bmatrix}$$

□

We can see that the numbers we calculated in Example 7.27 for the populations after the 10^{th} unit of time are not far from the long term values.

Consider another example.

Example 7.30: Populations After a Long Time

Suppose a migration matrix is given by

$$A = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

Find the comparison between the populations in the three locations after a long time.

Solution. In order to compare the populations in the long term, we want to find the steady state vector X_s . Solve

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} \frac{4}{5} & -\frac{1}{2} & -\frac{1}{5} & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ -\frac{11}{20} & -\frac{1}{4} & \frac{7}{10} & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{16}{19} & 0 \\ 0 & 1 & -\frac{18}{19} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so an eigenvector is

$$\begin{bmatrix} 16 \\ 18 \\ 19 \end{bmatrix}$$

Therefore, the proportion of population in location 2 to location 1 is given by $\frac{18}{16}$. The proportion of population 3 to location 2 is given by $\frac{19}{18}$.

□

Eigenvalues of Markov Matrices

The following is an important proposition.

Proposition 7.31: Eigenvalues of a Migration Matrix

Let $A = [a_{ij}]$ be a migration matrix. Then 1 is always an eigenvalue for A .

Proof. Remember that the determinant of a matrix always equals that of its transpose. Therefore,

$$\det(xI - A) = \det((xI - A)^T) = \det(xI - A^T)$$

because $I^T = I$. Thus the characteristic equation for A is the same as the characteristic equation for A^T . Consequently, A and A^T have the same eigenvalues. We will show that 1 is an eigenvalue for A^T and then it will follow that 1 is an eigenvalue for A .

Remember that for a migration matrix, $\sum_i a_{ij} = 1$. Therefore, if $A^T = [b_{ij}]$ with $b_{ij} = a_{ji}$, it follows that

$$\sum_j b_{ij} = \sum_j a_{ji} = 1$$

Therefore, from matrix multiplication,

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_j b_{ij} \\ \vdots \\ \sum_j b_{ij} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Notice that this shows that $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector for A^T corresponding to the eigenvalue, $\lambda = 1$. As explained above, this shows that $\lambda = 1$ is an eigenvalue for A because A and A^T have the same eigenvalues. \square

7.3.4. DYNAMICAL SYSTEMS

The migration matrices discussed above give an example of a discrete dynamical system. We call them discrete because they involve discrete values taken at a sequence of points rather than on a continuous interval of time.

An example of a situation which can be studied in this way is a predator prey model. Consider the following model where x is the number of prey and y the number of predators in a certain area at a certain time. These are functions of $n \in \mathbb{N}$ where $n = 1, 2, \dots$ are the ends of intervals of time which may be of interest in the problem. In other words, $x(n)$ is the number of prey at the end of the n^{th} interval of time. An example of this situation may be modeled by the following equation

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

This says that from time period n to $n+1$, x increases if there are more x and decreases as there are more y . In the context of this example, this means that as the number of predators increases, the number of prey decreases. As for y , it increases if there are more y and also if there are more x .

This is an example of a matrix recurrence which we define now.

Definition 7.32: Matrix Recurrence

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

This system can be expressed as $V_{n+1} = AV_n$ where $V_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

In this section, we will examine how to find solutions to a dynamical system given certain initial conditions. This process involves several concepts previously studied, including matrix

diagonalization and Markov matrices. The procedure is given as follows. Recall that when diagonalized, we can write $A^n = PD^nP^{-1}$.

Procedure 7.33: Solving a Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned}x_{n+1} &= ax_n + by_n \\y_{n+1} &= cx_n + dy_n\end{aligned}$$

Given initial conditions x_0 and y_0 , the solutions to the system are found as follows:

1. Express the dynamical system in the form $V_{n+1} = AV_n$.
2. Diagonalize A to be written as $A = PDP^{-1}$.
3. Then $V_n = PD^nP^{-1}V_0$ where V_0 is the vector containing the initial conditions.
4. If given specific values for n , substitute into this equation. Otherwise, find a general solution for n .

We will now consider an example in detail.

Example 7.34: Solutions of a Discrete Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned}x_{n+1} &= 1.5x_n - 0.5y_n \\y_{n+1} &= 1.0x_n\end{aligned}$$

Express this system as a matrix recurrence and find solutions to the dynamical system for initial conditions $x_0 = 20, y_0 = 10$.

Solution. First, we express the system as a matrix recurrence.

$$\begin{aligned}V_{n+1} &= AV_n \\ \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} &= \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}\end{aligned}$$

Then

$$A = \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix}$$

You can verify that the eigenvalues of A are 1 and .5. By diagonalizing, we can write A in the form

$$P^{-1}DP = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Now given an initial condition

$$V_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

the solution to the dynamical system is given by

$$\begin{aligned} V_n &= PD^n P^{-1} V_0 \\ \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}^n \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (.5)^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} y_0 ((.5)^n - 1) - x_0 ((.5)^n - 2) \\ y_0 (2(.5)^n - 1) - x_0 (2(.5)^n - 2) \end{bmatrix} \end{aligned}$$

If we let n become arbitrarily large, this vector approaches

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Thus for large n ,

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \approx \begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Now suppose the initial condition is given by

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Then, we can find solutions for various values of n . Here are the solutions for values of n between 1 and 5

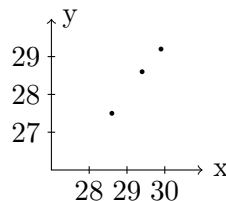
$$n = 1 : \begin{bmatrix} 25.0 \\ 20.0 \end{bmatrix}, n = 2 : \begin{bmatrix} 27.5 \\ 25.0 \end{bmatrix}, n = 3 : \begin{bmatrix} 28.75 \\ 27.5 \end{bmatrix}$$

$$n = 4 : \begin{bmatrix} 29.375 \\ 28.75 \end{bmatrix}, n = 5 : \begin{bmatrix} 29.688 \\ 29.375 \end{bmatrix}$$

Notice that as n increases, we approach the vector given by

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix} = \begin{bmatrix} 2(20) - 10 \\ 2(20) - 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

These solutions are graphed in the following figure.



□

The following example demonstrates another system which exhibits some interesting behavior. When we graph the solutions, it is possible for the ordered pairs to spiral around the origin.

Example 7.35: Finding Solutions to a Dynamical System

Suppose a dynamical system is of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Find solutions to the dynamical system for given initial conditions.

Solution. Let

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$$

To find solutions, we must diagonalize A . You can verify that the eigenvalues of A are complex and are given by $\lambda_1 = .7 + .7i$ and $\lambda_2 = .7 - .7i$. The eigenvector for $\lambda_1 = .7 + .7i$ is

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

and that the eigenvector for $\lambda_2 = .7 - .7i$ is

$$\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus the matrix A can be written in the form

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} .7 + .7i & 0 \\ 0 & .7 - .7i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

and so,

$$\begin{aligned} V_n &= PD^nP^{-1}V_0 \\ \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} (.7 + .7i)^n & 0 \\ 0 & (.7 - .7i)^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \end{aligned}$$

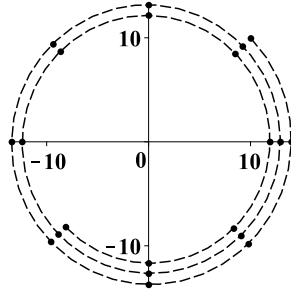
The explicit solution is given by

$$\begin{bmatrix} x_0 \left(\frac{1}{2} (0.7 - 0.7i)^n + \frac{1}{2} (0.7 + 0.7i)^n \right) + y_0 \left(\frac{1}{2}i (0.7 - 0.7i)^n - \frac{1}{2}i (0.7 + 0.7i)^n \right) \\ y_0 \left(\frac{1}{2} (0.7 - 0.7i)^n + \frac{1}{2} (0.7 + 0.7i)^n \right) - x_0 \left(\frac{1}{2}i (0.7 - 0.7i)^n - \frac{1}{2}i (0.7 + 0.7i)^n \right) \end{bmatrix}$$

Suppose the initial condition is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Then one obtains the following sequence of values which are graphed below by letting $n = 1, 2, \dots, 20$



In this picture, the dots are the values and the dashed line is to help to picture what is happening.

These points are getting gradually closer to the origin, but they are circling the origin in the clockwise direction as they do so. As n increases, the vector $\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ \square

This type of behavior along with complex eigenvalues is typical of the deviations from an equilibrium point in the Lotka Volterra system of differential equations which is a famous model for predator-prey interactions. These differential equations are given by

$$\begin{aligned} x' &= x(a - by) \\ y' &= -y(c - dx) \end{aligned}$$

where a, b, c, d are positive constants. For example, you might have X be the population of moose and Y the population of wolves on an island.

Note that these equations make logical sense. The top says that the rate at which the moose population increases would be aX if there were no predators Y . However, this is modified by multiplying instead by $(a - bY)$ because if there are predators, these will militate against the population of moose. The more predators there are, the more pronounced is this effect. As to the predator equation, you can see that the equations predict that if there are many prey around, then the rate of growth of the predators would seem to be high. However, this is modified by the term $-cY$ because if there are many predators, there would be competition for the available food supply and this would tend to decrease Y' .

The behavior near an equilibrium point, which is a point where the right side of the differential equations equals zero, is of great interest. In this case, the equilibrium point is

$$x = \frac{c}{d}, y = \frac{a}{b}$$

Then one defines new variables according to the formula

$$x + \frac{c}{d} = x, \quad y = y + \frac{a}{b}$$

In terms of these new variables, the differential equations become

$$\begin{aligned}x' &= \left(x + \frac{c}{d}\right) \left(a - b\left(y + \frac{a}{b}\right)\right) \\y' &= -\left(y + \frac{a}{b}\right) \left(c - d\left(x + \frac{c}{d}\right)\right)\end{aligned}$$

Multiplying out the right sides yields

$$\begin{aligned}x' &= -bxy - b\frac{c}{d}y \\y' &= dxy + \frac{a}{b}dx\end{aligned}$$

The interest is for x, y small and so these equations are essentially equal to

$$x' = -b\frac{c}{d}y, \quad y' = \frac{a}{b}dx$$

Replace x' with the difference quotient $\frac{x(t+h)-x(t)}{h}$ where h is a small positive number and y' with a similar difference quotient. For example one could have h correspond to one day or even one hour. Thus, for h small enough, the following would seem to be a good approximation to the differential equations.

$$\begin{aligned}x(t+h) &= x(t) - hb\frac{c}{d}y \\y(t+h) &= y(t) + h\frac{a}{b}dx\end{aligned}$$

Let $1, 2, 3, \dots$ denote the ends of discrete intervals of time having length h chosen above. Then the above equations take the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Note that the eigenvalues of this matrix are always complex.

We are not interested in time intervals of length h for h very small. Instead, we are interested in much longer lengths of time. Thus, replacing the time interval with mh ,

$$\begin{bmatrix} x(n+m) \\ y(n+m) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix}^m \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

For example, if $m = 2$, you would have

$$\begin{bmatrix} x(n+2) \\ y(n+2) \end{bmatrix} = \begin{bmatrix} 1 - ach^2 & -2b\frac{c}{d}h \\ 2\frac{a}{b}dh & 1 - ach^2 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Note that most of the time, the eigenvalues of the new matrix will be complex.

You can also notice that the upper right corner will be negative by considering higher powers of the matrix. Thus letting $1, 2, 3, \dots$ denote the ends of discrete intervals of time, the desired discrete dynamical system is of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a & -b \\ c & d \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

where a, b, c, d are positive constants and the matrix will likely have complex eigenvalues because it is a power of a matrix which has complex eigenvalues.

You can see from the above discussion that if the eigenvalues of the matrix used to define the dynamical system are less than 1 in absolute value, then the origin is stable in the sense that as $n \rightarrow \infty$, the solution converges to the origin. If either eigenvalue is larger than 1 in absolute value, then the solutions to the dynamical system will usually be unbounded, unless the initial condition is chosen very carefully. The next example exhibits the case where one eigenvalue is larger than 1 and the other is smaller than 1.

The following example demonstrates a familiar concept as a dynamical system.

Example 7.36: The Fibonacci Sequence

The Fibonacci sequence is the sequence given by

$$1, 1, 2, 3, 5, \dots$$

which is defined recursively in the form

$$x(0) = 1 = x(1), \quad x(n+2) = x(n+1) + x(n)$$

Show how the Fibonacci Sequence can be considered a dynamical system.

Solution. This sequence is extremely important in the study of reproducing rabbits. It can be considered as a dynamical system as follows. Let $y(n) = x(n+1)$. Then the above recurrence relation can be written as

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{1}{2}$. The corresponding eigenvectors are, respectively,

$$X_1 = \begin{bmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}$$

You can see from a short computation that one of the eigenvalues is smaller than 1 in absolute value while the other is larger than 1 in absolute value. Now, diagonalizing A gives

us

$$\begin{aligned} & \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\sqrt{5} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix} \end{aligned}$$

Then it follows that for a given initial condition, the solution to this dynamical system is of the form

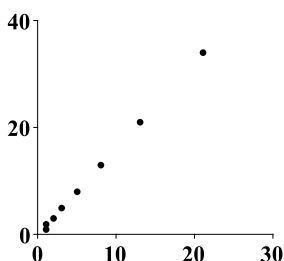
$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{1}{10}\sqrt{5} + \frac{1}{2} \\ -\frac{1}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It follows that

$$x(n) = \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n \left(\frac{1}{10}\sqrt{5} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}\right)$$

□

Here is a picture of the ordered pairs $(x(n), y(n))$ for $n = 0, 1, \dots, n$.



There is so much more that can be said about dynamical systems. It is a major topic of study in differential equations and what is given above is just an introduction.

7.3.5. EXERCISES

1. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Diagonalize A to find A^{10} .

2. Let $A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. Diagonalize A to find A^{50} .

3. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. Diagonalize A to find A^{100} .

4. The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{7}{10} & \frac{1}{9} & \frac{1}{5} \\ \frac{1}{10} & \frac{7}{9} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{9} & \frac{2}{5} \end{bmatrix}$$

- (a) Initially, there are 90 people in location 1, 81 in location 2, and 85 in location 3. How many are in each location after one time period?
- (b) The total number of individuals in the migration process is 256. After a long time, how many are in each location?

5. The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

- (a) Initially, there are 130 individuals in location 1, 300 in location 2, and 70 in location 3. How many are in each location after two time periods?
- (b) The total number of individuals in the migration process is 500. After a long time, how many are in each location?

6. The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{1}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{3}{5} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

The total number of individuals in the migration process is 480. After a long time, how many are in each location?

7. The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{1}{3} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{3} & \frac{7}{10} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{10} \end{bmatrix}$$

The total number of individuals in the migration process is 1155. After a long time, how many are in each location?

8. The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{2}{5} & \frac{1}{10} & \frac{1}{8} \\ \frac{3}{10} & \frac{2}{5} & \frac{5}{8} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

The total number of individuals in the migration process is 704. After a long time, how many are in each location?

9. You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{3}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at *NE* ends in *NW* is $\frac{1}{10}$, the probability that a trailer starting at *SW* ends in *NW* is $\frac{1}{5}$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 413?

10. You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the

following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at *NE* ends in *NW* is $1/10$, the probability that a trailer starting at *SW* ends in *NW* is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 1469.

11. The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

12. The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{10}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{4}{9}$
p.c.	$\frac{7}{10}$	$\frac{2}{5}$	$\frac{2}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

13. You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{5}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{11}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{11}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at *NE* ends in *NW* is $1/10$, the probability that a trailer starting at *SW* ends in *NW* is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 407?

14. The University of Poohbah offers three degree programs, scouting education (SE), dance appreciation (DA), and engineering (E). It has been determined that the probabilities of transferring from one program to another are as in the following table.

	SE	DA	E
SE	.8	.1	.3
DA	.1	.7	.5
E	.1	.2	.2

where the number indicates the probability of transferring from the top program to the program on the left. Thus the probability of going from DA to E is .2. Find the probability that a student is enrolled in the various programs.

15. In the city of Nabal, there are three political persuasions, republicans (R), democrats (D), and neither one (N). The following table shows the transition probabilities between the political parties, the top row being the initial political party and the side row being the political affiliation the following year.

	R	D	N
R	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{2}{7}$
D	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{7}$
N	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

Find the probabilities that a person will be identified with the various political persuasions. Which party will end up being most important?

16. The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{7}$ and so forth.

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{5}{9}$
sunny	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{1}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

7.4 ORTHOGONALITY

7.4.1. ORTHOGONAL DIAGONALIZATION

We begin this section by recalling some important definitions. Recall from Definition 4.93 that non-zero vectors are called orthogonal if their dot product equals 0. A set is orthonormal if it is orthogonal and each vector is a unit vector.

An orthogonal matrix U , from Definition 4.97, is one in which $UU^T = I$. In other words, the transpose of an orthogonal matrix is equal to its inverse. A key characteristic of orthogonal matrices, which will be essential in this section, is that the columns of an orthogonal matrix form an orthonormal set.

We now recall another important definition.

Definition 7.37: Symmetric and Skew Symmetric Matrices

A real $n \times n$ matrix A , is **symmetric** if $A^T = A$. If $A = -A^T$, then A is called **skew symmetric**.

Before proving an essential theorem, we first examine the following lemma which will be used below.

Lemma 7.38: The Dot Product

Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$A\vec{x} \bullet \vec{y} = \vec{x} \bullet A\vec{y}$$

Proof. This result follows from the definition of the dot product together with properties of matrix multiplication, as follows:

$$\begin{aligned}
A\vec{x} \bullet \vec{y} &= \sum_{k,l} a_{kl} x_l y_k \\
&= \sum_{k,l} (a_{lk})^T x_l y_k \\
&= \vec{x} \bullet A^T \vec{y} \\
&= \vec{x} \bullet A \vec{y}
\end{aligned}$$

The last step follows from $A^T = A$, since A is symmetric. \square

We can now prove that the eigenvalues of a real symmetric matrix are real numbers. Consider the following important theorem.

Theorem 7.39: Orthogonal Eigenvectors

Let A be a real symmetric matrix. Then the eigenvalues of A are real numbers and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Recall that for a complex number $a + ib$, the complex conjugate, denoted by $\overline{a + ib}$ is given by $\overline{a + ib} = a - ib$. The notation, $\overline{\vec{x}}$ will denote the vector which has every entry replaced by its complex conjugate.

Suppose A is a real symmetric matrix and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T \vec{x} = (\overline{A\vec{x}})^T \vec{x} = \overline{\vec{x}}^T A^T \vec{x} = \overline{\vec{x}}^T A \vec{x} = \lambda \overline{\vec{x}}^T \vec{x}$$

Dividing by $\overline{\vec{x}}^T \vec{x}$ on both sides yields $\overline{\lambda} = \lambda$ which says λ is real. To do this, we need to ensure that $\overline{\vec{x}}^T \vec{x} \neq 0$. Notice that $\overline{\vec{x}}^T \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. Since we chose \vec{x} such that $A\vec{x} = \lambda\vec{x}$, \vec{x} is an eigenvector and therefore must be nonzero.

Now suppose A is real symmetric and $A\vec{x} = \lambda\vec{x}$, $A\vec{y} = \mu\vec{y}$ where $\mu \neq \lambda$. Then since A is symmetric, it follows from Lemma 7.38 about the dot product that

$$\lambda\vec{x} \bullet \vec{y} = A\vec{x} \bullet \vec{y} = \vec{x} \bullet A\vec{y} = \vec{x} \bullet \mu\vec{y} = \mu\vec{x} \bullet \vec{y}$$

Hence $(\lambda - \mu) \vec{x} \bullet \vec{y} = 0$. It follows that, since $\lambda - \mu \neq 0$, it must be that $\vec{x} \bullet \vec{y} = 0$. Therefore the eigenvectors form an orthogonal set. \square

The following theorem is proved in a similar manner.

Theorem 7.40: Eigenvalues of Skew Symmetric Matrix

The eigenvalues of a real skew symmetric matrix are either equal to 0 or are pure imaginary numbers.

Proof. First, note that if $A = 0$ is the zero matrix, then A is skew symmetric and has eigenvalues equal to 0.

Suppose $A = -A^T$ so A is skew symmetric and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T \vec{x} = (\overline{A\vec{x}})^T \vec{x} = \overline{\vec{x}}^T A^T \vec{x} = -\overline{\vec{x}}^T A\vec{x} = -\lambda\overline{\vec{x}}^T \vec{x}$$

and so, dividing by $\overline{\vec{x}}^T \vec{x}$ as before, $\overline{\lambda} = -\lambda$. Letting $\lambda = a + ib$, this means $a - ib = -a - ib$ and so $a = 0$. Thus λ is pure imaginary. \square

Consider the following example.

Example 7.41: Eigenvalues of a Skew Symmetric Matrix

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find its eigenvalues.

Solution. First notice that A is skew symmetric. By Theorem 7.40, the eigenvalues will either equal 0 or be pure imaginary. The eigenvalues of A are obtained by solving the usual equation

$$\det(xI - A) = \det \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = x^2 + 1 = 0$$

Hence the eigenvalues are $\pm i$, pure imaginary. \square

Consider the following example.

Example 7.42: Eigenvalues of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find its eigenvalues.

Solution. First, notice that A is symmetric. By Theorem 7.39, the eigenvalues will all be real. The eigenvalues of A are obtained by solving the usual equation

$$\det(xI - A) = \det \begin{bmatrix} x-1 & -2 \\ -2 & x-3 \end{bmatrix} = x^2 - 4x - 1 = 0$$

The eigenvalues are given by $\lambda_1 = 2 + \sqrt{5}$ and $\lambda_2 = 2 - \sqrt{5}$ which are both real. \square

Recall that a diagonal matrix $D = [d_{ij}]$ is one in which $d_{ij} = 0$ whenever $i \neq j$. In other words, all numbers not on the main diagonal are equal to zero.

Consider the following important theorem.

Theorem 7.43: Orthogonal Diagonalization

Let A be a real symmetric matrix. Then there exists an orthogonal matrix U such that

$$U^T A U = D$$

where D is a diagonal matrix. Moreover, the diagonal entries of D are the eigenvalues of A .

We can use this theorem to diagonalize a symmetric matrix, using orthogonal matrices. Consider the following corollary.

Corollary 7.44: Orthonormal Set of Eigenvectors

If A is a real $n \times n$ symmetric matrix, then there exists an orthonormal set of eigenvectors, $\{\vec{u}_1, \dots, \vec{u}_n\}$.

Proof. Since A is symmetric, then by Theorem 7.43, there exists an orthogonal matrix U such that $U^T A U = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A . Therefore, since A is symmetric and all the matrices are real,

$$\overline{D} = \overline{D^T} = \overline{U^T A^T U} = U^T A^T U = U^T A U = D$$

showing D is real because each entry of D equals its complex conjugate.

Now let

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$

where the \vec{u}_i denote the columns of U and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

The equation, $U^T A U = D$ implies $A U = U D$ and

$$\begin{aligned} A U &= \begin{bmatrix} A \vec{u}_1 & A \vec{u}_2 & \cdots & A \vec{u}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \vec{u}_1 & \lambda_2 \vec{u}_2 & \cdots & \lambda_n \vec{u}_n \end{bmatrix} \\ &= U D \end{aligned}$$

where the entries denote the columns of $A U$ and $U D$ respectively. Therefore, $A \vec{u}_i = \lambda_i \vec{u}_i$. Since the matrix U is orthogonal, the ij^{th} entry of $U^T U$ equals δ_{ij} and so

$$\delta_{ij} = \vec{u}_i^T \vec{u}_j = \vec{u}_i \bullet \vec{u}_j$$

This proves the corollary because it shows the vectors $\{\vec{u}_i\}$ form an orthonormal set. \square

Example 7.45: Find an Orthonormal Set of Eigenvectors

Find an orthonormal set of eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 6 & 4 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution. Recall Procedure 7.5 for finding the eigenvalues and eigenvectors of a matrix. You can verify that the eigenvalues are 18, 9, 2. First find the eigenvector for 18 by solving the equation $(18I - A)X = 0$. The appropriate augmented matrix is given by

$$\left[\begin{array}{ccc|c} 18-17 & 2 & 2 & 0 \\ 2 & 18-6 & -4 & 0 \\ 2 & -4 & 18-6 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore an eigenvector is

$$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 9$. The augmented matrix and resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 9-17 & 2 & 2 & 0 \\ 2 & 9-6 & -4 & 0 \\ 2 & -4 & 9-6 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Thus an eigenvector for $\lambda = 9$ is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Finally find an eigenvector for $\lambda = 2$. The appropriate augmented matrix and reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 2-17 & 2 & 2 & 0 \\ 2 & 2-6 & -4 & 0 \\ 2 & -4 & 2-6 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Thus an eigenvector for $\lambda = 2$ is

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The set of eigenvectors for A is given by

$$\left\{ \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

You can verify that these eigenvectors form an orthogonal set. By dividing each eigenvector by its magnitude, we obtain an orthonormal set:

$$\left\{ \frac{1}{\sqrt{18}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

□

Consider the following example.

Example 7.46: Repeated Eigenvalues

Find an orthonormal set of three eigenvectors for the matrix

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Solution. You can verify that the eigenvalues of A are 9 (with multiplicity two) and 18 (with multiplicity one). Consider the eigenvectors corresponding to $\lambda = 9$. The appropriate augmented matrix and reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 9-10 & -2 & -2 & 0 \\ -2 & 9-13 & -4 & 0 \\ -2 & -4 & 9-13 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so eigenvectors are of the form

$$\begin{bmatrix} -2y - 2z \\ y \\ z \end{bmatrix}$$

We need to find two of these which are orthogonal. Let one be given by setting $z = 0$ and $y = 1$, giving $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In order to find an eigenvector orthogonal to this one, we need to satisfy

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2y - 2z \\ y \\ z \end{bmatrix} = 5y + 4z = 0$$

The values $y = -4$ and $z = 5$ satisfy this equation, giving another eigenvector corresponding to $\lambda = 9$ as

$$\begin{bmatrix} -2(-4) - 2(5) \\ (-4) \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 18$. The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 18-10 & -2 & -2 & 0 \\ -2 & 18-13 & -4 & 0 \\ -2 & -4 & 18-13 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so an eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Dividing each eigenvector by its length, the orthonormal set is

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{5}}{15} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

□

In the above solution, the repeated eigenvalue implies that there would have been many other orthonormal bases which could have been obtained. While we chose to take $z = 0, y = 1$, we could just as easily have taken $y = 0$ or even $y = z = 1$. Any such change would have resulted in a different orthonormal set.

Recall the following definition.

Definition 7.47: Diagonalizable

An $n \times n$ matrix A is said to be **non defective** or **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

As indicated in Theorem 7.43 if A is a real symmetric matrix, there exists an orthogonal matrix U such that $U^T A U = D$ where D is a diagonal matrix. Therefore, every symmetric matrix is diagonalizable because if U is an orthogonal matrix, it is invertible and its inverse is U^T . In this case, we say that A is **orthogonally diagonalizable**. In the following example, this orthogonal matrix U will be found.

Example 7.48: Diagonalize a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix U such that $U^T A U$ is a diagonal matrix.

Solution. In this case, the eigenvalues are 2 (with multiplicity one) and 1 (with multiplicity two). First we will find an eigenvector for the eigenvalue 2. The appropriate augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

However, it is desired that the eigenvectors be unit vectors and so dividing this vector by its length gives

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next find the eigenvectors corresponding to the eigenvalue equal to 1. The appropriate augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Two of these which are orthonormal are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, choosing $s = 1$ and $t = 0$, and $\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$,

letting $s = 0$, $t = 1$ and normalizing the resulting vector.

To obtain the desired orthogonal matrix, we let the orthonormal eigenvectors computed above be the columns.

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To verify, compute $U^T A U$ as follows:

$$\begin{aligned} U^T A U &= \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D \end{aligned}$$

the desired diagonal matrix. Notice that the eigenvectors, which construct the columns of U , are in the same order as the eigenvalues in D . \square

7.4.2. POSITIVE DEFINITE MATRICES

7.4.3. QR FACTORIZATION

In this section, a reliable factorization of matrices is studied. Called the QR factorization of a matrix, it *always* exists. While much can be said about the QR factorization, this section will be limited to real matrices. Therefore we assume the dot product used below is the usual dot product. We begin with a definition.

Definition 7.49: QR Factorization

Let A be a real $m \times n$ matrix. Then a QR factorization of A consists of two matrices, Q orthogonal and R upper triangular, such that $A = QR$.

The following theorem claims that such a factorization exists.

Theorem 7.50: Existence of QR Factorization

Let A be any real $m \times n$ matrix with linearly independent columns. Then there exists an orthogonal matrix Q and an upper triangular matrix R having non-negative entries on the main diagonal such that

$$A = QR$$

The procedure for obtaining the QR factorization for any matrix A is as follows.

Procedure 7.51: QR Factorization

Let A be an $m \times n$ matrix given by $A = [A_1 \ A_2 \ \cdots \ A_n]$ where the A_i are the linearly independent columns of A .

1. Apply the Gram-Schmidt Process 4.102 to the columns of A , writing B_i for the resulting columns.
2. Normalize the B_i , to find $C_i = \frac{1}{\|B_i\|} B_i$.
3. Construct the orthogonal matrix Q as $Q = [C_1 \ C_2 \ \cdots \ C_n]$.
4. Construct the upper triangular matrix R as

$$R = \begin{bmatrix} \|B_1\| & A_2 \bullet C_1 & A_3 \bullet C_1 & \cdots & A_n \bullet C_1 \\ 0 & \|B_2\| & A_3 \bullet C_2 & \cdots & A_n \bullet C_2 \\ 0 & 0 & \|B_3\| & \cdots & A_n \bullet C_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|B_n\| \end{bmatrix}$$

5. Finally, write $A = QR$ where Q is the orthogonal matrix and R is the upper triangular matrix obtained above.

Notice that Q is an orthogonal matrix as the C_i form an orthonormal set. Since $\|B_i\| > 0$ for all i (since the length of a vector is always positive), it follows that R is an upper triangular matrix with positive entries on the main diagonal.

Consider the following example.

Example 7.52: Finding a QR Factorization

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find an orthogonal matrix Q and upper triangular matrix R such that $A = QR$.

Solution. First, observe that A_1, A_2 , the columns of A , are linearly independent. Therefore we can use the Gram-Schmidt Process to create a corresponding orthogonal set $\{B_1, B_2\}$ as

follows:

$$\begin{aligned}
 B_1 &= A_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
 B_2 &= A_2 - \frac{A_2 \bullet B_1}{\|B_1\|^2} B_1 \\
 &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
 \end{aligned}$$

Normalize each vector to create the set $\{C_1, C_2\}$ as follows:

$$\begin{aligned}
 C_1 &= \frac{1}{\|B_1\|} B_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
 C_2 &= \frac{1}{\|B_2\|} B_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
 \end{aligned}$$

Now construct the orthogonal matrix Q as

$$\begin{aligned}
 Q &= [C_1 \ C_2 \ \cdots \ C_n] \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}
 \end{aligned}$$

Finally, construct the upper triangular matrix R as

$$\begin{aligned}
 R &= \begin{bmatrix} \|B_1\| & A_2 \bullet C_1 \\ 0 & \|B_2\| \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}
 \end{aligned}$$

It is left to the reader to verify that $A = QR$. □

The QR Factorization and Eigenvalues

The QR factorization of a matrix has a very useful application. It turns out that it can be used repeatedly to estimate the eigenvalues of a matrix. Consider the following procedure.

Procedure 7.53: Using the QR Factorization to Estimate Eigenvalues

Let A be an invertible matrix. Define the matrices A_1, A_2, \dots as follows:

1. $A_1 = A$ factored as $A_1 = Q_1 R_1$
2. $A_2 = R_1 Q_1$ factored as $A_2 = Q_2 R_2$
3. $A_3 = R_2 Q_2$ factored as $A_3 = Q_3 R_3$

Continue in this manner, where in general $A_k = Q_k R_k$ and $A_{k+1} = R_k Q_k$.

Then it follows that this sequence of A_i converges to an upper triangular matrix which is similar to A . Therefore the eigenvalues of A can be approximated by the entries on the main diagonal of this upper triangular matrix.

7.4.4. QUADRATIC FORMS

One of the applications of orthogonal diagonalization is that of quadratic forms and graphs of level curves of a quadratic form. This section has to do with rotation of axes so that with respect to the new axes, the graph of the level curve of a quadratic form is oriented parallel to the coordinate axes. This makes it much easier to understand. For example, we all know that $x_1^2 + x_2^2 = 1$ represents the equation in two variables whose graph in \mathbb{R}^2 is a circle of radius 1. But how do we know what the graph of the equation $5x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ represents?

We first formally define what is meant by a quadratic form. In this section we will work with only *real* quadratic forms, which means that the coefficients will all be real numbers.

Definition 7.54: Quadratic Form

A **quadratic form** is a polynomial of degree two in n variables x_1, x_2, \dots, x_n , written as a linear combination of x_i^2 terms and $x_i x_j$ terms.

Consider the quadratic form $q = a_{11}x_1^2 + a_{22}x_2^2 + \dots + a_{nn}x_n^2 + a_{12}x_1x_2 + \dots$. We can write

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the vector whose entries are the variables contained in the quadratic form.

Similarly, let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be the matrix whose entries are the coefficients of

x_i^2 and $x_i x_j$ from q . Note that the matrix A is not unique, and we will consider this further in the example below. Using this matrix A , the quadratic form can be written as $q = \vec{x}^T A \vec{x}$.

$$\begin{aligned}
q &= \vec{x}^T A \vec{x} \\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \\
&= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots
\end{aligned}$$

Let's explore how to find this matrix A . Consider the following example.

Example 7.55: Matrix of a Quadratic Form

Let a quadratic form q be given by

$$q = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Write q in the form $\vec{x}^T A \vec{x}$.

Solution. First, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then, writing $q = \vec{x}^T A \vec{x}$ gives

$$\begin{aligned}
q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2
\end{aligned}$$

Notice that we have an x_1x_2 term as well as an x_2x_1 term. Since multiplication is commutative, these terms can be combined. This means that q can be written

$$q = a_{11}x_1^2 + (a_{21} + a_{12})x_1x_2 + a_{22}x_2^2$$

Equating this to q as given in the example, we have

$$a_{11}x_1^2 + (a_{21} + a_{12})x_1x_2 + a_{22}x_2^2 = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Therefore,

$$\begin{aligned}
a_{11} &= 6 \\
a_{22} &= 3 \\
a_{21} + a_{12} &= 4
\end{aligned}$$

This demonstrates that the matrix A is not unique, as there are several correct solutions to $a_{21} + a_{12} = 4$. However, we will *always* choose the coefficients such that $a_{21} = a_{12} = \frac{1}{2}(a_{21} + a_{12})$. This results in $a_{21} = a_{12} = 2$. This choice is key, as it will ensure that A turns out to be a symmetric matrix.

Hence,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

You can verify that $q = \vec{x}^T A \vec{x}$ holds for this choice of A . □

The above procedure for choosing A to be symmetric applies for any quadratic form q . We will *always* choose coefficients such that $a_{ij} = a_{ji}$.

We now turn our attention to the focus of this section. Our goal is to start with a quadratic form q as given above and find a way to rewrite it to eliminate the $x_i x_j$ terms. This is done through a change of variables. In other words, we wish to find y_i such that

$$q = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$$

Letting $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $D = [d_{ij}]$, we can write $q = \vec{y}^T D \vec{y}$ where D is the matrix of

coefficients from q . There is something special about this matrix D that is crucial. Since no $y_i y_j$ terms exist in q , it follows that $d_{ij} = 0$ for all $i \neq j$. Therefore, D is a diagonal matrix. Through this change of variables, we find the **principal axes** y_1, y_2, \dots, y_n of the quadratic form.

This discussion sets the stage for the following essential theorem.

Theorem 7.56: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables x_1, \dots, x_n . It follows that q can be written in the form $q = \vec{x}^T A \vec{x}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $A = [a_{ij}]$ is the symmetric matrix of coefficients of q .

New variables y_1, y_2, \dots, y_n can be found such that $q = \vec{y}^T D \vec{y}$ where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and $D = [d_{ij}]$ is a diagonal matrix. The matrix D contains the eigenvalues of A and is found by orthogonally diagonalizing A .

While not a formal proof, the following discussion should convince you that the above theorem holds. Let q be a quadratic form in the variables x_1, \dots, x_n . Then, q can be written in the form $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A . By Theorem 7.43 we can orthogonally diagonalize the matrix A such that $U^T A U = D$ for an orthogonal matrix U and diagonal matrix D .

Then, the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is found by $\vec{y} = U^T \vec{x}$. To see that this works, rewrite $\vec{y} = U^T \vec{x}$ as $\vec{x} = U \vec{y}$. Letting $q = \vec{x}^T A \vec{x}$, proceed as follows:

$$\begin{aligned} q &= \vec{x}^T A \vec{x} \\ &= (U \vec{y})^T A (U \vec{y}) \\ &= \vec{y}^T (U^T A U) \vec{y} \\ &= \vec{y}^T D \vec{y} \end{aligned}$$

The following procedure details the steps for the change of variables given in the above theorem.

Procedure 7.57: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables x_1, \dots, x_n given by

$$q = a_{11}x_1^2 + a_{22}x_2^2 + \dots + a_{nn}x_n^2 + a_{12}x_1x_2 + \dots$$

Then, q can be written as $q = d_{11}y_1^2 + \dots + d_{nn}y_n^2$ as follows:

1. Write $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A .
2. Orthogonally diagonalize A to be written as $U^T A U = D$ for an orthogonal matrix U and diagonal matrix D .

$$3. \text{ Write } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \text{ Then, } \vec{x} = U \vec{y}.$$

4. The quadratic form q will now be given by

$$q = d_{11}y_1^2 + \dots + d_{nn}y_n^2 = \vec{y}^T D \vec{y}$$

where $D = [d_{ij}]$ is the diagonal matrix found by orthogonally diagonalizing A .

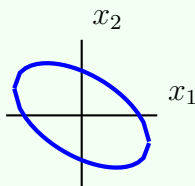
Consider the following example.

Example 7.58: Choosing New Axes to Simplify a Quadratic Form

Consider the following level curve

$$6x_1^2 + 4x_1x_2 + 3x_2^2 = 7$$

shown in the following graph.



Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution. Notice that the level curve is given by $q = 7$ for $q = 6x_1^2 + 4x_1x_2 + 3x_2^2$. This is the same quadratic form that we examined earlier in Example 7.55. Therefore we know that we can write $q = \vec{x}^T A \vec{x}$ for the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

Now we want to orthogonally diagonalize A to write $U^T A U = D$ for an orthogonal matrix U and diagonal matrix D . The details are left to the reader, and you can verify that the resulting matrices are

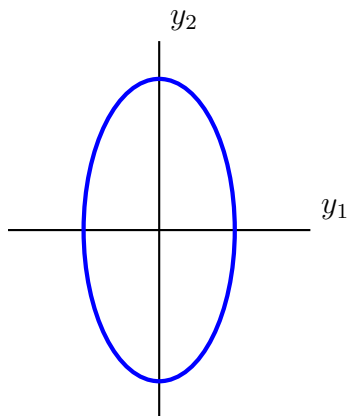
$$U = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$
$$D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

Next we write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. It follows that $\vec{x} = U\vec{y}$.

We can now express the quadratic form q in terms of y , using the entries from D as coefficients as follows:

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 7y_1^2 + 2y_2^2 \end{aligned}$$

Hence the level curve can be written $7y_1^2 + 2y_2^2 = 7$. The graph of this equation is given by:



The change of variables results in new axes such that with respect to the new axes, the ellipse is oriented parallel to the coordinate axes. These are called the **principal axes** of the quadratic form. \square

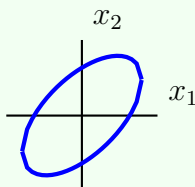
The following is another example of diagonalizing a quadratic form.

Example 7.59: Choosing New Axes to Simplify a Quadratic Form

Consider the level curve

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = 8$$

shown in the following graph.



Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution. First, express the level curve as $\vec{x}^T A \vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is symmetric. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $q = \vec{x}^T A \vec{x}$ is given by

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \end{aligned}$$

Equating this to the given description for q , we have

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

This implies that $a_{11} = 5, a_{22} = 5$ and in order for A to be symmetric, $a_{12} = a_{21} = \frac{1}{2}(a_{12} + a_{21}) = -3$. The result is $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. We can write $q = \vec{x}^T A \vec{x}$ as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8$$

Next, orthogonally diagonalize the matrix A to write $U^T A U = D$. The details are left to the reader and the necessary matrices are given by

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

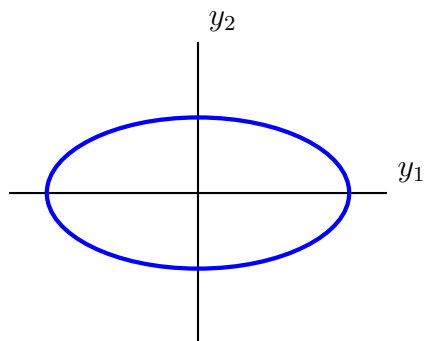
Write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, such that $\vec{x} = U\vec{y}$. Then it follows that q is given by

$$q = d_{11}y_1^2 + d_{22}y_2^2$$

$$= 2y_1^2 + 8y_2^2$$

Therefore the level curve can be written as $2y_1^2 + 8y_2^2 = 8$.

This is an ellipse which is parallel to the coordinate axes. Its graph is of the form



Thus this change of variables chooses new axes such that with respect to these new axes, the ellipse is oriented parallel to the coordinate axes. \square

7.4.5. EXERCISES

1. Find the eigenvalues and an orthonormal basis of eigenvectors for A .

$$A = \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

2. Find the eigenvalues and an orthonormal basis of eigenvectors for A .

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Hint: One eigenvalue is 3.

3. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Hint: One eigenvalue is -2.

4. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 18 and 24.

5. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

6. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} -\frac{5}{3} & \frac{1}{15}\sqrt{6}\sqrt{5} & \frac{8}{15}\sqrt{5} \\ \frac{1}{15}\sqrt{6}\sqrt{5} & -\frac{14}{5} & -\frac{1}{15}\sqrt{6} \\ \frac{8}{15}\sqrt{5} & -\frac{1}{15}\sqrt{6} & \frac{7}{15} \end{bmatrix}$$

Hint: The eigenvalues are $-3, -2, 1$.

7. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

8. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

9. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} \frac{4}{3} & \frac{1}{3}\sqrt{3}\sqrt{2} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3}\sqrt{2} & 1 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{5}{3} \end{bmatrix}$$

Hint: The eigenvalues are 0, 2, 2 where 2 is listed twice because it is a root of multiplicity 2.

10. Find the eigenvalues and an orthonormal basis of eigenvectors for A . Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T A U = D$.

$$A = \begin{bmatrix} 1 & \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & \frac{1}{12}\sqrt{2}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{6} & \frac{1}{12}\sqrt{2}\sqrt{6} & \frac{1}{2} \end{bmatrix}$$

Hint: The eigenvalues are 2, 1, 0.

11. Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6}\sqrt{3}\sqrt{2} & -\frac{7}{18}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & -\frac{1}{12}\sqrt{2}\sqrt{6} \\ -\frac{7}{18}\sqrt{3}\sqrt{6} & -\frac{1}{12}\sqrt{2}\sqrt{6} & -\frac{5}{6} \end{bmatrix}$$

Hint: The eigenvalues are 1, 2, -2.

12. Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{bmatrix}$$

Hint: The eigenvalues are $-1, 2, -1$ where -1 is listed twice because it has multiplicity 2 as a zero of the characteristic equation.

13. Explain why a matrix A is symmetric if and only if there exists an orthogonal matrix U such that $A = U^T D U$ for D a diagonal matrix.
14. Show that if A is a real symmetric matrix and λ and μ are two different eigenvalues, then if \vec{x} is an eigenvector for λ and \vec{y} is an eigenvector for μ , then $\vec{x} \bullet \vec{y} = 0$. Also all eigenvalues are real. Supply reasons for each step in the following argument. First

$$\lambda \vec{x}^T \vec{x} = (A\vec{x})^T \vec{x} = \vec{x}^T A\vec{x} = \vec{x}^T \overline{A\vec{x}} = \vec{x}^T \overline{\lambda \vec{x}} = \overline{\lambda} \vec{x}^T \vec{x}$$

and so $\lambda = \overline{\lambda}$. This shows that all eigenvalues are real. It follows all the eigenvectors are real. Why? Now let \vec{x}, \vec{y}, μ and λ be given as above.

$$\lambda (\vec{x} \bullet \vec{y}) = \lambda \vec{x} \bullet \vec{y} = A\vec{x} \bullet \vec{y} = \vec{x} \bullet A\vec{y} = \vec{x} \bullet \mu \vec{y} = \mu (\vec{x} \bullet \vec{y}) = \mu (\vec{x} \bullet \vec{y})$$

and so

$$(\lambda - \mu) \vec{x} \bullet \vec{y} = 0$$

Why does it follow that $\vec{x} \bullet \vec{y} = 0$?

15. Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

16. Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$\text{span} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

17. A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A is a symmetric matrix.

18. Given a quadratic form in three variables, x, y , and z , show there exists an orthogonal matrix U and variables x', y', z' such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

with the property that in terms of the new variables, the quadratic form is

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \lambda_3 (z')^2$$

where the numbers, λ_1, λ_2 , and λ_3 are the eigenvalues of the matrix A in Problem 17.

19. Consider the quadratic form q given by $q = 3x_1^2 - 12x_1x_2 - 2x_2^2$.
- (a) Write q in the form $\vec{x}^T A \vec{x}$ for an appropriate symmetric matrix A .
 - (b) Use a change of variables to rewrite q to eliminate the x_1x_2 term.
20. Consider the quadratic form q given by $q = -2x_1^2 + 2x_1x_2 - 2x_2^2$.
- (a) Write q in the form $\vec{x}^T A \vec{x}$ for an appropriate symmetric matrix A .
 - (b) Use a change of variables to rewrite q to eliminate the x_1x_2 term.
21. Consider the quadratic form q given by $q = 7x_1^2 + 6x_1x_2 - x_2^2$.
- (a) Write q in the form $\vec{x}^T A \vec{x}$ for an appropriate symmetric matrix A .
 - (b) Use a change of variables to rewrite q to eliminate the x_1x_2 term.

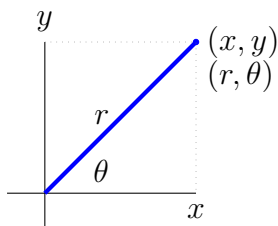
8. SOME CURVILINEAR COORDINATE SYSTEMS

8.1 POLAR COORDINATES AND POLAR GRAPHS

Outcomes

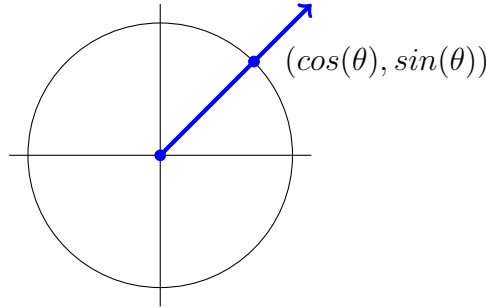
- A. Understand polar coordinates.
- B. Convert points between Cartesian and polar coordinates.

You have likely encountered the Cartesian coordinate system in many aspects of mathematics. There is an alternative way to represent points in space, called **polar coordinates**. The idea is suggested in the following picture.



Consider the point above, which would be specified as (x, y) in Cartesian coordinates. We can also specify this point using polar coordinates, which we write as (r, θ) . The number r is the distance from the origin $(0, 0)$ to the point, while θ is the angle shown between the positive x axis and the line from the origin to the point. In this way, the point can be specified in polar coordinates as (r, θ) .

Now suppose we are given an ordered pair (r, θ) where r and θ are real numbers. We want to determine the point specified by this ordered pair. We can use θ to identify a ray from the origin as follows. Let the ray pass from $(0, 0)$ through the point $(\cos \theta, \sin \theta)$ as shown.



The ray is identified on the graph as the line from the origin, through the point $(\cos(\theta), \sin(\theta))$. Now if $r > 0$, go a distance equal to r in the direction of the displayed arrow starting at $(0, 0)$. If $r < 0$, move in the opposite direction a distance of $|r|$. This is the point determined by (r, θ) .

It is common to assume that θ is in the interval $[0, 2\pi)$ and $r > 0$. In this case, there is a very simple relationship between the Cartesian and polar coordinates, given by

$$x = r \cos(\theta), \quad y = r \sin(\theta) \quad (8.1)$$

These equations demonstrate how to find the Cartesian coordinates when we are given the polar coordinates of a point. They can also be used to find the polar coordinates when we know (x, y) . A simpler way to do this is the following equations:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan(\theta) &= \frac{y}{x} \end{aligned} \quad (8.2)$$

In the next example, we look at how to find the Cartesian coordinates of a point specified by polar coordinates.

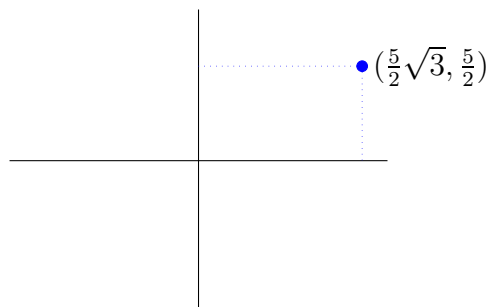
Example 8.1: Finding Cartesian Coordinates

The polar coordinates of a point in the plane are $(5, \pi/6)$. Find the Cartesian coordinates of this point.

Solution. The point is specified by the polar coordinates $(5, \pi/6)$. Therefore $r = 5$ and $\theta = \pi/6$. From [8.1](#)

$$\begin{aligned} x &= r \cos(\theta) = 5 \cos\left(\frac{\pi}{6}\right) = \frac{5}{2}\sqrt{3} \\ y &= r \sin(\theta) = 5 \sin\left(\frac{\pi}{6}\right) = \frac{5}{2} \end{aligned}$$

Thus the Cartesian coordinates are $(\frac{5}{2}\sqrt{3}, \frac{5}{2})$. The point is shown in the below graph.



□

Consider the following example of the case where $r < 0$.

Example 8.2: Finding Cartesian Coordinates

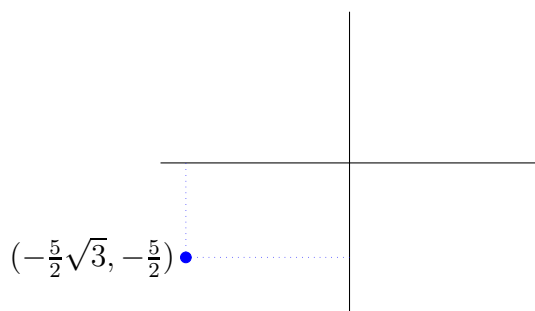
The polar coordinates of a point in the plane are $(-5, \pi/6)$. Find the Cartesian coordinates.

Solution. For the point specified by the polar coordinates $(-5, \pi/6)$, $r = -5$, and $\theta = \pi/6$. From 8.1

$$x = r \cos(\theta) = -5 \cos\left(\frac{\pi}{6}\right) = -\frac{5}{2}\sqrt{3}$$

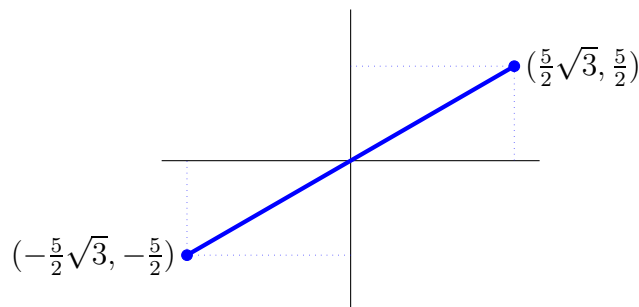
$$y = r \sin(\theta) = -5 \sin\left(\frac{\pi}{6}\right) = -\frac{5}{2}$$

Thus the Cartesian coordinates are $(-\frac{5}{2}\sqrt{3}, -\frac{5}{2})$. The point is shown in the following graph.



Recall from the previous example that for the point specified by $(5, \pi/6)$, the Cartesian coordinates are $(\frac{5}{2}\sqrt{3}, \frac{5}{2})$. Notice that in this example, by multiplying r by -1 , the resulting Cartesian coordinates are also multiplied by -1 . □

The following picture exhibits both points in the above two examples to emphasize how they are just on opposite sides of $(0, 0)$ but at the same distance from $(0, 0)$.



In the next two examples, we look at how to convert Cartesian coordinates to polar coordinates.

Example 8.3: Finding Polar Coordinates

Suppose the Cartesian coordinates of a point are $(3, 4)$. Find a pair of polar coordinates which correspond to this point.

Solution. Using equation 8.2, we can find r and θ . Hence $r = \sqrt{3^2 + 4^2} = 5$. It remains to identify the angle θ between the positive x axis and the line from the origin to the point. Since both the x and y values are positive, the point is in the first quadrant. Therefore, θ is between 0 and $\pi/2$. Using this and 8.2, we have to solve:

$$\tan(\theta) = \frac{4}{3}$$

Conversely, we can use equation 8.1 as follows:

$$3 = 5 \cos(\theta)$$

$$4 = 5 \sin(\theta)$$

Solving these equations, we find that, approximately, $\theta = 0.927\,295$ radians. □

Consider the following example.

Example 8.4: Finding Polar Coordinates

Suppose the Cartesian coordinates of a point are $(-\sqrt{3}, 1)$. Find the polar coordinates which correspond to this point.

Solution. Given the point $(-\sqrt{3}, 1)$,

$$\begin{aligned} r &= \sqrt{1^2 + (-\sqrt{3})^2} \\ &= \sqrt{1 + 3} \\ &= 2 \end{aligned}$$

In this case, the point is in the second quadrant since the x value is negative and the y value is positive. Therefore, θ will be between $\pi/2$ and π . Solving the equations

$$-\sqrt{3} = 2 \cos(\theta)$$

$$1 = 2 \sin(\theta)$$

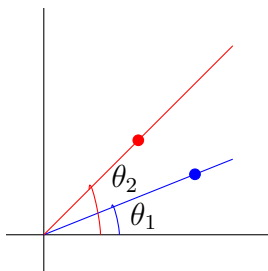
we find that $\theta = 5\pi/6$. Hence the polar coordinates for this point are $(2, 5\pi/6)$. \square

Consider this example. Suppose we used $r = -2$ and $\theta = 2\pi - (\pi/6) = 11\pi/6$. These coordinates specify the same point as above. Observe that there are infinitely many ways to identify this particular point with polar coordinates. In fact, every point can be represented with polar coordinates in infinitely many ways. Because of this, it will usually be the case that θ is confined to lie in some interval of length 2π and $r > 0$, for real numbers r and θ .

Just as with Cartesian coordinates, it is possible to use relations between the polar coordinates to specify points in the plane. The process of sketching the graphs of these relations is very similar to that used to sketch graphs of functions in Cartesian coordinates. Consider a relation between polar coordinates of the form, $r = f(\theta)$. To graph such a relation, first make a table of the form

θ	r
θ_1	$f(\theta_1)$
θ_2	$f(\theta_2)$
\vdots	\vdots

Graph the resulting points and connect them with a curve. The following picture illustrates how to begin this process.



To find the point in the plane corresponding to the ordered pair $(f(\theta), \theta)$, we follow the same process as when finding the point corresponding to (r, θ) .

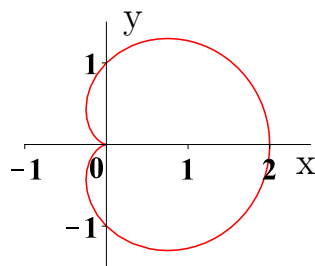
Consider the following example of this procedure, incorporating computer software.

Example 8.5: Graphing a Polar Equation

Graph the polar equation $r = 1 + \cos \theta$.

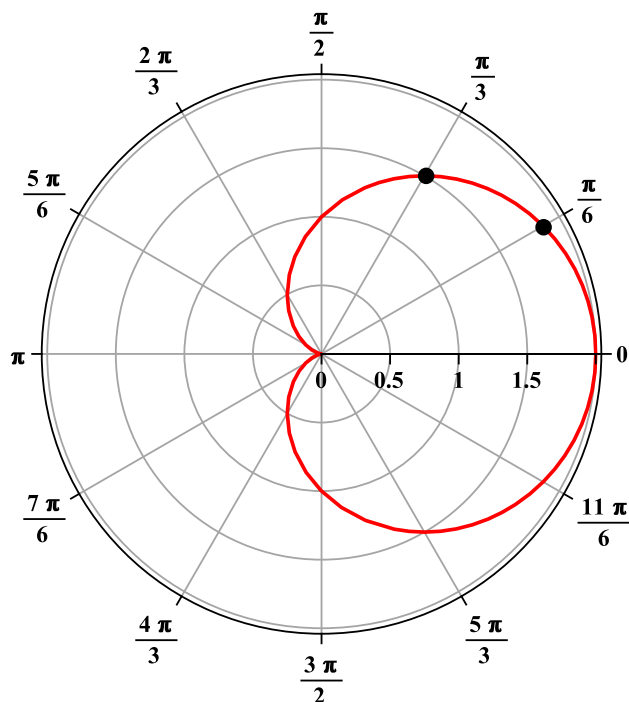
Solution. We will use the computer software *Maple* to complete this example. The command which produces the polar graph of the above equation is: `> plot(1+cos(t),t=0..2*Pi,coords=polar).`

Here we use t to represent the variable θ for convenience. The command tells Maple that r is given by $1 + \cos(t)$ and that $t \in [0, 2\pi]$.



The above graph makes sense when considered in terms of trigonometric functions. Suppose $\theta = 0, r = 2$ and let θ increase to $\pi/2$. As θ increases, $\cos \theta$ decreases to 0. Thus the line from the origin to the point on the curve should get shorter as θ goes from 0 to $\pi/2$. As θ goes from $\pi/2$ to π , $\cos \theta$ decreases, eventually equaling -1 at $\theta = \pi$. Thus $r = 0$ at this point. This scenario is depicted in the above graph, which shows a function called a **cardioid**.

The following picture illustrates the above procedure for obtaining the polar graph of $r = 1 + \cos(\theta)$. In this picture, the concentric circles correspond to values of r while the rays from the origin correspond to the angles which are shown on the picture. The dot on the ray corresponding to the angle $\pi/6$ is located at a distance of $r = 1 + \cos(\pi/6)$ from the origin. The dot on the ray corresponding to the angle $\pi/3$ is located at a distance of $r = 1 + \cos(\pi/3)$ from the origin and so forth. The polar graph is obtained by connecting such points with a smooth curve, with the result being the figure shown above.



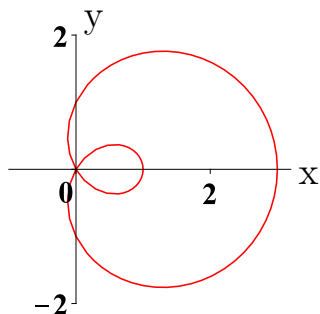
□

Consider another example of constructing a polar graph.

Example 8.6: A Polar Graph

Graph $r = 1 + 2 \cos \theta$ for $\theta \in [0, 2\pi]$.

Solution. The graph of the polar equation $r = 1 + 2 \cos \theta$ for $\theta \in [0, 2\pi]$ is given as follows.

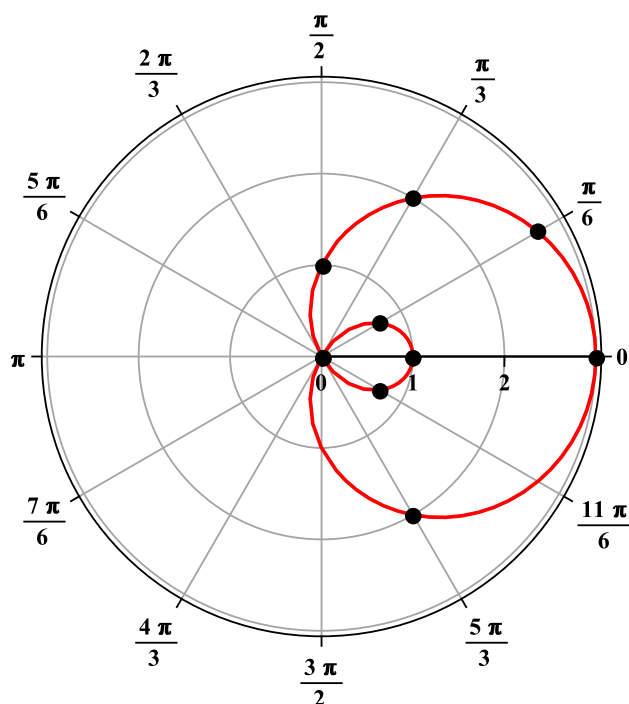


To see the way this is graphed, consider the following picture. First the indicated points were graphed and then the curve was drawn to connect the points. When done by a computer, many more points are used to create a more accurate picture.

Consider first the following table of points.

θ	$\pi/6$	$\pi/3$	$\pi/2$	$5\pi/6$	π	$4\pi/3$	$7\pi/6$	$5\pi/3$
r	$\sqrt{3} + 1$	2	1	$1 - \sqrt{3}$	-1	0	$1 - \sqrt{3}$	2

Note how some entries in the table have $r < 0$. To graph these points, simply move in the opposite direction. These types of points are responsible for the small loop on the inside of the larger loop in the graph.



□

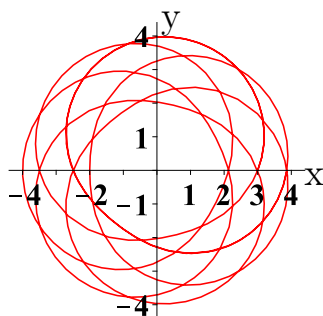
The process of constructing these graphs can be greatly facilitated by computer software. However, the use of such software should not replace understanding the steps involved.

The next example shows the graph for the equation $r = 3 + \sin\left(\frac{7\theta}{6}\right)$. For complicated polar graphs, computer software is used to facilitate the process.

Example 8.7: A Polar Graph

Graph $r = 3 + \sin\left(\frac{7\theta}{6}\right)$ for $\theta \in [0, 14\pi]$.

Solution.



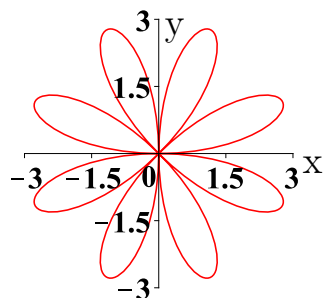
□

The next example shows another situation in which r can be negative.

Example 8.8: A Polar Graph: Negative r

Graph $r = 3\sin(4\theta)$ for $\theta \in [0, 2\pi]$.

Solution.



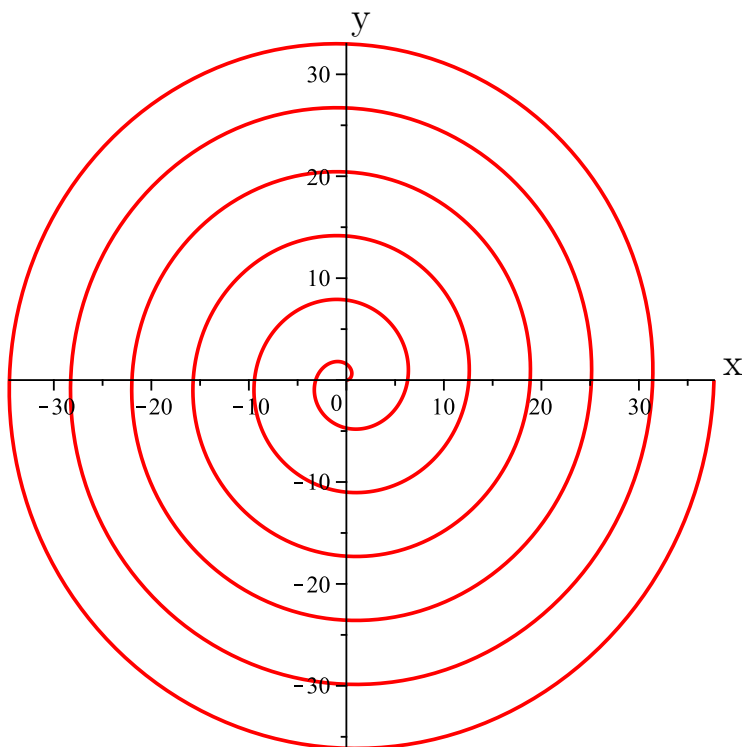
□

We conclude this section with an interesting graph of a simple polar equation.

Example 8.9: The Graph of a Spiral

Graph $r = \theta$ for $\theta \in [0, 2\pi]$.

Solution. The graph of this polar equation is a spiral. This is the case because as θ increases, so does r .



□

In the next section, we will look at two ways of generalizing polar coordinates to three dimensions.

8.1.1. EXERCISES

1. In the following, polar coordinates (r, θ) for a point in the plane are given. Find the corresponding Cartesian coordinates.

- (a) $(2, \pi/4)$
- (b) $(-2, \pi/4)$
- (c) $(3, \pi/3)$
- (d) $(-3, \pi/3)$
- (e) $(2, 5\pi/6)$
- (f) $(-2, 11\pi/6)$
- (g) $(2, \pi/2)$
- (h) $(1, 3\pi/2)$
- (i) $(-3, 3\pi/4)$
- (j) $(3, 5\pi/4)$
- (k) $(-2, \pi/6)$

2. Consider the following Cartesian coordinates (x, y) . Find polar coordinates corresponding to these points.

- (a) $(-1, 1)$
- (b) $(\sqrt{3}, -1)$
- (c) $(0, 2)$
- (d) $(-5, 0)$
- (e) $(-2\sqrt{3}, 2)$
- (f) $(2, -2)$
- (g) $(-1, \sqrt{3})$
- (h) $(-1, -\sqrt{3})$

3. The following relations are written in terms of Cartesian coordinates (x, y) . Rewrite them in terms of polar coordinates, (r, θ) .

- (a) $y = x^2$
- (b) $y = 2x + 6$
- (c) $x^2 + y^2 = 4$
- (d) $x^2 - y^2 = 1$

4. Use a calculator or computer algebra system to graph the following polar relations.

- (a) $r = 1 - \sin(2\theta), \theta \in [0, 2\pi]$
 - (b) $r = \sin(4\theta), \theta \in [0, 2\pi]$
 - (c) $r = \cos(3\theta) + \sin(2\theta), \theta \in [0, 2\pi]$
 - (d) $r = \theta, \theta \in [0, 15]$
5. Graph the polar equation $r = 1 + \sin \theta$ for $\theta \in [0, 2\pi]$.
 6. Graph the polar equation $r = 2 + \sin \theta$ for $\theta \in [0, 2\pi]$.
 7. Graph the polar equation $r = 1 + 2 \sin \theta$ for $\theta \in [0, 2\pi]$.
 8. Graph the polar equation $r = 2 + \sin(2\theta)$ for $\theta \in [0, 2\pi]$.
 9. Graph the polar equation $r = 1 + \sin(2\theta)$ for $\theta \in [0, 2\pi]$.
 10. Graph the polar equation $r = 1 + \sin(3\theta)$ for $\theta \in [0, 2\pi]$.
 11. Describe how to solve for r and θ in terms of x and y in polar coordinates.
 12. This problem deals with parabolas, ellipses, and hyperbolas and their equations. Let $l, e > 0$ and consider

$$r = \frac{l}{1 \pm e \cos \theta}$$

Show that if $e = 0$, the graph of this equation gives a circle. Show that if $0 < e < 1$, the graph is an ellipse, if $e = 1$ it is a parabola and if $e > 1$, it is a hyperbola.

8.2 SPHERICAL AND CYLINDRICAL COORDINATES

Outcomes

- A. Understand cylindrical and spherical coordinates.
- B. Convert points between Cartesian, cylindrical, and spherical coordinates.

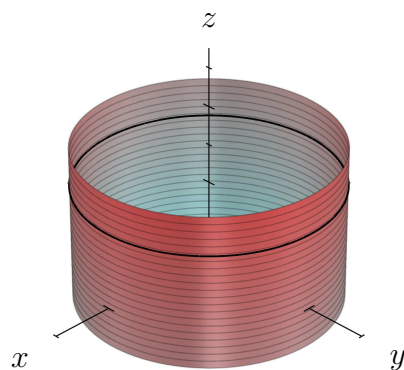
Spherical and cylindrical coordinates are two generalizations of polar coordinates to three dimensions. We will first look at **cylindrical coordinates**.

When moving from polar coordinates in two dimensions to cylindrical coordinates in three dimensions, we use the polar coordinates in the xy plane and add a z coordinate. For this reason, we use the notation (r, θ, z) to express cylindrical coordinates. The relationship between Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) is given by

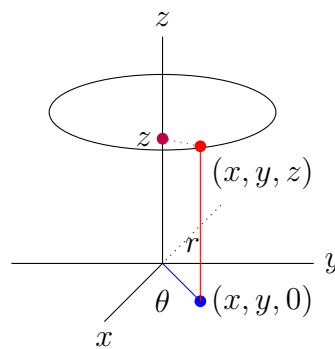
$$\begin{aligned}x &= r \cos(\theta) \\y &= r \sin(\theta) \\z &= z\end{aligned}$$

where $r \geq 0$, $\theta \in [0, 2\pi)$, and z is simply the Cartesian coordinate. Notice that x and y are defined as the usual polar coordinates in the xy -plane. Recall that r is defined as the length of the ray from the origin to the point $(x, y, 0)$, while θ is the angle between the positive x -axis and this same ray.

To illustrate this coordinate system, consider the following two pictures. In the first of these, both r and z are known. The cylinder corresponds to a given value for r . A useful way to think of r is as the distance between a point in three dimensions and the z -axis. Every point on the cylinder shown is at the same distance from the z -axis. Giving a value for z results in a horizontal circle, or cross section of the cylinder at the given height on the z axis (shown below as a black line on the cylinder). In the second picture, the point is specified completely by also knowing θ as shown.



r and z are known



r , θ and z are known

Every point of three dimensional space other than the z axis has unique cylindrical coordinates. Of course there are infinitely many cylindrical coordinates for the origin and for the z -axis. Any θ will work if $r = 0$ and z is given.

Consider now **spherical coordinates**, the second generalization of polar form in three dimensions. For a point (x, y, z) in three dimensional space, the spherical coordinates are defined as follows.

ρ : the length of the ray from the origin to the point

θ : the angle between the positive x -axis and the ray from the origin to the point $(x, y, 0)$

ϕ : the angle between the positive z -axis and the ray from the origin to the point of interest

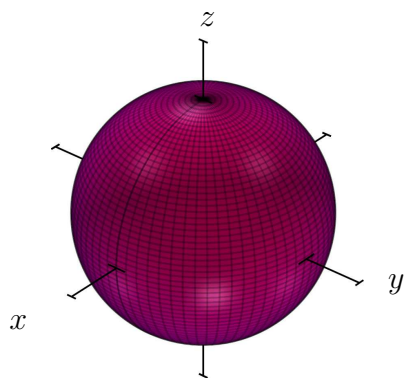
The spherical coordinates are determined by (ρ, ϕ, θ) . The relation between these and the Cartesian coordinates (x, y, z) for a point are as follows.

$$x = \rho \sin(\phi) \cos(\theta), \quad \phi \in [0, \pi]$$

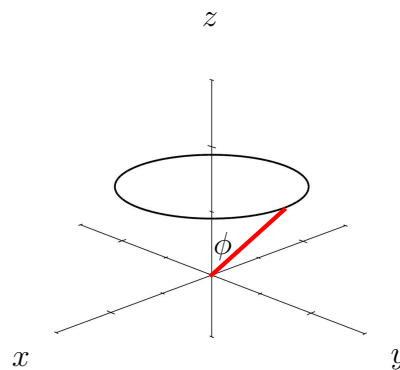
$$y = \rho \sin(\phi) \sin(\theta), \quad \theta \in [0, 2\pi)$$

$$z = \rho \cos \phi, \quad \rho \geq 0.$$

Consider the pictures below. The first illustrates the surface when ρ is known, which is a sphere of radius ρ . The second picture corresponds to knowing both ρ and ϕ , which results in a circle about the z -axis. Suppose the first picture demonstrates a graph of the Earth. Then the circle in the second picture would correspond to a particular latitude.

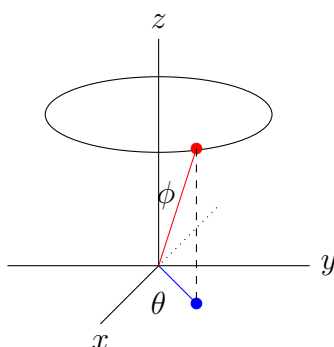


ρ is known



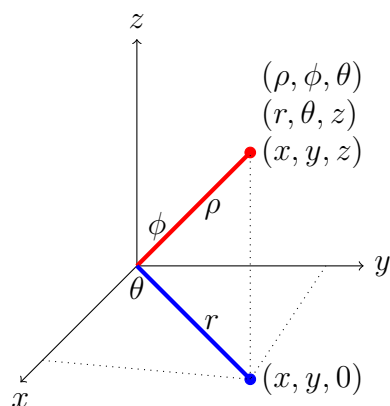
ρ and ϕ are known

Giving the third coordinate, θ completely specifies the point of interest. This is demonstrated in the following picture. If the latitude corresponds to ϕ , then we can think of θ as the longitude.



ρ , ϕ and θ are known

The following picture summarizes the geometric meaning of the three coordinate systems.



Therefore, we can represent the same point in three ways, using Cartesian coordinates, (x, y, z) , cylindrical coordinates, (r, θ, z) , and spherical coordinates (ρ, ϕ, θ) .

Using this picture to review, call the point of interest P for convenience. The Cartesian coordinates for P are (x, y, z) . Then ρ is the distance between the origin and the point P . The angle between the positive z axis and the line between the origin and P is denoted by ϕ . Then θ is the angle between the positive x axis and the line joining the origin to the point $(x, y, 0)$ as shown. This gives the spherical coordinates, (ρ, ϕ, θ) . Given the line from the origin to $(x, y, 0)$, $r = \rho \sin(\phi)$ is the length of this line. Thus r and θ determine a point in the xy -plane. In other words, r and θ are the usual polar coordinates and $r \geq 0$ and $\theta \in [0, 2\pi)$. Letting z denote the usual z coordinate of a point in three dimensions, (r, θ, z) are the cylindrical coordinates of P .

The relation between spherical and cylindrical coordinates is that $r = \rho \sin(\phi)$ and the θ is the same as the θ of cylindrical and polar coordinates.

We will now consider some examples.

Example 8.10: Describing a Surface in Spherical Coordinates

Express the surface $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$ in spherical coordinates.

Solution. We will use the equations from above:

$$\begin{aligned}x &= \rho \sin(\phi) \cos(\theta), \phi \in [0, \pi] \\y &= \rho \sin(\phi) \sin(\theta), \theta \in [0, 2\pi) \\z &= \rho \cos \phi, \rho \geq 0\end{aligned}$$

To express the surface in spherical coordinates, we substitute these expressions into the equation. This is done as follows:

$$\rho \cos(\phi) = \frac{1}{\sqrt{3}} \sqrt{(\rho \sin(\phi) \cos(\theta))^2 + (\rho \sin(\phi) \sin(\theta))^2} = \frac{1}{3} \sqrt{3} \rho \sin(\phi).$$

This reduces to

$$\tan(\phi) = \sqrt{3}$$

and so $\phi = \pi/3$. □

Example 8.11: Describing a Surface in Spherical Coordinates

Express the surface $y = x$ in terms of spherical coordinates.

Solution. Using the same procedure as the previous example, this says $\rho \sin(\phi) \sin(\theta) = \rho \sin(\phi) \cos(\theta)$. Simplifying, $\sin(\theta) = \cos(\theta)$, which you could also write $\tan(\theta) = 1$. □

We conclude this section with an example of how to describe a surface using cylindrical coordinates.

Example 8.12: Describing a Surface in Cylindrical Coordinates

Express the surface $x^2 + y^2 = 4$ in cylindrical coordinates.

Solution. Recall that to convert from Cartesian to cylindrical coordinates, we can use the following equations:

$$x = r \cos(\theta), y = r \sin(\theta), z = z$$

Substituting these equations in for x, y, z in the equation for the surface, we have

$$r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = 4$$

This can be written as $r^2(\cos^2(\theta) + \sin^2(\theta)) = 4$. Recall that $\cos^2(\theta) + \sin^2(\theta) = 1$. Thus $r^2 = 4$ or $r = 2$. \square

8.2.1. EXERCISES

1. The following are the cylindrical coordinates of points, (r, θ, z) . Find the Cartesian and spherical coordinates of each point.

- (a) $(5, \frac{5\pi}{6}, -3)$
- (b) $(3, \frac{\pi}{3}, 4)$
- (c) $(4, \frac{2\pi}{3}, 1)$
- (d) $(2, \frac{3\pi}{4}, -2)$
- (e) $(3, \frac{3\pi}{2}, -1)$
- (f) $(8, \frac{11\pi}{6}, -11)$

2. The following are the Cartesian coordinates of points, (x, y, z) . Find the cylindrical and spherical coordinates of these points.

- (a) $(\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}, -3)$
- (b) $(\frac{3}{2}, \frac{3}{2}\sqrt{3}, 2)$
- (c) $(-\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}, 11)$
- (d) $(-\frac{5}{2}, \frac{5}{2}\sqrt{3}, 23)$
- (e) $(-\sqrt{3}, -1, -5)$
- (f) $(\frac{3}{2}, -\frac{3}{2}\sqrt{3}, -7)$
- (g) $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$
- (h) $(-\frac{1}{2}\sqrt{3}, \frac{3}{2}, 1)$
- (i) $(-\frac{3}{4}\sqrt{2}, \frac{3}{4}\sqrt{2}, -\frac{3}{2}\sqrt{3})$

(j) $(-\sqrt{3}, 1, 2\sqrt{3})$

(k) $(-\frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{6}, -\frac{1}{2}\sqrt{2})$

3. The following are spherical coordinates of points in the form (ρ, ϕ, θ) . Find the Cartesian and cylindrical coordinates of each point.

(a) $(4, \frac{\pi}{4}, \frac{5\pi}{6})$

(b) $(2, \frac{\pi}{3}, \frac{2\pi}{3})$

(c) $(3, \frac{5\pi}{6}, \frac{3\pi}{2})$

(d) $(4, \frac{\pi}{2}, \frac{7\pi}{4})$

(e) $(4, \frac{2\pi}{3}, \frac{\pi}{6})$

(f) $(4, \frac{3\pi}{4}, \frac{5\pi}{3})$

4. Describe the surface $\phi = \pi/4$ in Cartesian coordinates, where ϕ is the polar angle in spherical coordinates.
5. Describe the surface $\theta = \pi/4$ in spherical coordinates, where θ is the angle measured from the positive x axis.
6. Describe the surface $r = 5$ in Cartesian coordinates, where r is one of the cylindrical coordinates.
7. Describe the surface $\rho = 4$ in Cartesian coordinates, where ρ is the distance to the origin.
8. Give the cone described by $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates and in spherical coordinates.
9. The following are described in Cartesian coordinates. Rewrite them in terms of spherical coordinates.

(a) $z = x^2 + y^2$.

(b) $x^2 - y^2 = 1$.

(c) $z^2 + x^2 + y^2 = 6$.

(d) $z = \sqrt{x^2 + y^2}$.

(e) $y = x$.

(f) $z = x$.

10. The following are described in Cartesian coordinates. Rewrite them in terms of cylindrical coordinates.

(a) $z = x^2 + y^2$.

(b) $x^2 - y^2 = 1$.

(c) $z^2 + x^2 + y^2 = 6.$

(d) $z = \sqrt{x^2 + y^2}.$

(e) $y = x.$

(f) $z = x.$

9. VECTOR SPACES

9.1 ALGEBRAIC CONSIDERATIONS

Outcomes

- A. *Develop the abstract concept of a vector space through axioms.*
- B. *Deduce basic properties of vector spaces.*
- C. *Use the vector space axioms to determine if a set and its operations constitute a vector space.*

In this section we consider the idea of an abstract vector space. A vector space is something which has two operations satisfying the following vector space axioms. In the following definition we define two operations; vector addition, denoted by $+$ and scalar multiplication denoted by placing the scalar next to the vector. A vector space need not have usual operations, and for this reason the operations will always be given in the definition of the vector space. The below axioms for addition (written $+$) and scalar multiplication must hold for however addition and scalar multiplication are defined for the vector space.

It is important to note that we have seen much of this content before, in terms of \mathbb{R}^n . We will prove in this section that \mathbb{R}^n is an example of a vector space and therefore all discussions in this chapter will pertain to \mathbb{R}^n . While it may be useful to consider all concepts of this chapter in terms of \mathbb{R}^n , it is also important to understand that these concepts apply to *all* vector spaces.

In the following definition, we will choose scalars a, b to be real numbers and are thus dealing with *real* vector spaces. However, we could also choose scalars which are complex numbers. In this case, we would call the vector space V complex.

Definition 9.1: Vector Space Axioms

A vector space V is a set of vectors with two operations defined, addition and scalar multiplication. Let $\vec{v}, \vec{w}, \vec{z}$ be vectors in V . Then they satisfy the following axioms of addition:

- Closed under Addition

If \vec{v}, \vec{w} are in V , then $\vec{v} + \vec{w}$ is also in V .

- The Commutative Law of Addition

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

- The Associative Law of Addition

$$(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$$

- The Existence of an Additive Identity

$$\vec{v} + \vec{0} = \vec{v}$$

- The Existence of an Additive Inverse

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Let $a, b \in \mathbb{R}$. The following axioms apply to the operation of scalar multiplication.

- Closed under Scalar Multiplication

If a is a real number, and \vec{v} is in V , then $a\vec{v}$ is in V .

-

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$$

-

$$(a + b)\vec{v} = a\vec{v} + b\vec{v}$$

-

$$a(b\vec{v}) = (ab)\vec{v}$$

-

$$1\vec{v} = \vec{v}$$

Consider the following example, in which we prove that \mathbb{R}^n is in fact a vector space.

Example 9.2: \mathbb{R}^n

\mathbb{R}^n , under the usual operations of vector addition and scalar multiplication, is a vector space.

Solution. To show that \mathbb{R}^n is a vector space, we need to show that the above axioms hold. Let $\vec{x}, \vec{y}, \vec{z}$ be vectors in \mathbb{R}^n . We first prove the axioms for vector addition.

- To show that \mathbb{R}^n is closed under addition, we must show that for two vectors in \mathbb{R}^n their sum is also in \mathbb{R}^n . The sum $\vec{x} + \vec{y}$ is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

The sum is a vector with n entries, showing that it is in \mathbb{R}^n . Hence \mathbb{R}^n is closed under vector addition.

- To show that addition is commutative, consider the following:

$$\begin{aligned} \vec{x} + \vec{y} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \\ &= \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ \vdots \\ y_n + x_n \end{bmatrix} \\ &= \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \vec{y} + \vec{x} \end{aligned}$$

Hence addition of vectors in \mathbb{R}^n is commutative.

- We will show that addition of vectors in \mathbb{R}^n is associative in a similar way.

$$\begin{aligned}
(\vec{x} + \vec{y}) + \vec{z} &= \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \\ \vdots \\ (x_n + y_n) + z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \\ \vdots \\ x_n + (y_n + z_n) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \\ \vdots \\ y_n + z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \vec{x} + (\vec{y} + \vec{z})
\end{aligned}$$

Hence addition of vectors is associative.

- Next, we show the existence of an additive identity. Let $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$\begin{aligned}
\vec{x} + \vec{0} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \vec{x}
\end{aligned}$$

Hence the zero vector $\vec{0}$ is an additive identity.

- Next, we prove the existence of an additive inverse. Let $-\vec{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$.

$$\begin{aligned}
\vec{x} + (-\vec{x}) &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 - x_1 \\ x_2 - x_2 \\ \vdots \\ x_n - x_n \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \vec{0}
\end{aligned}$$

Hence $-\vec{x}$ is an additive inverse.

We now need to prove the axioms related to scalar multiplication. Let a, b be real numbers and let \vec{x}, \vec{y} be vectors in \mathbb{R}^n .

- We first show that \mathbb{R}^n is closed under scalar multiplication. To do so, we show that $a\vec{x}$ is also a vector with n entries.

$$a\vec{x} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

The vector $a\vec{x}$ is again a vector with n entries, showing that \mathbb{R}^n is closed under scalar multiplication.

- We wish to show that $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.

$$\begin{aligned} a(\vec{x} + \vec{y}) &= a \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) \\ &= a \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \\ &= \begin{bmatrix} a(x_1 + y_1) \\ a(x_2 + y_2) \\ \vdots \\ a(x_n + y_n) \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + ay_1 \\ ax_2 + ay_2 \\ \vdots \\ ax_n + ay_n \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix} \\ &= a\vec{x} + a\vec{y} \end{aligned}$$

- Next, we wish to show that $(a + b)\vec{x} = a\vec{x} + b\vec{x}$.

$$\begin{aligned}
(a+b)\vec{x} &= (a+b) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 \\ \vdots \\ ax_n + bx_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} \\
&= a\vec{x} + b\vec{x}
\end{aligned}$$

- We wish to show that $a(b\vec{x}) = (ab)\vec{x}$.

$$\begin{aligned}
 a(b\vec{x}) &= a \left(b \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) \\
 &= a \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} \\
 &= \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix} \\
 &= \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} \\
 &= (ab) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
 &= (ab)\vec{x}
 \end{aligned}$$

- Finally, we need to show that $1\vec{x} = \vec{x}$.

$$\begin{aligned}
 1\vec{x} &= 1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
 &= \begin{bmatrix} 1x_1 \\ 1x_2 \\ \vdots \\ 1x_n \end{bmatrix} \\
 &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
 &= \vec{x}
 \end{aligned}$$

By the above proofs, it is clear that \mathbb{R}^n satisfies the vector space axioms. Hence, \mathbb{R}^n is a vector space under the usual operations of vector addition and scalar multiplication. \square

We now consider some examples of vector spaces.

Example 9.3: Vector Space of Polynomials

Let \mathbb{P}_2 be the set of all polynomials of at most degree 2 as well as the zero polynomial. Define addition to be the standard addition of polynomials, and scalar multiplication the usual multiplication of a polynomial by a number. Then \mathbb{P}_2 is a vector space.

Solution. We can write \mathbb{P}_2 explicitly as

$$\mathbb{P}_2 = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } i\}$$

To show that \mathbb{P}_2 is a vector space, we verify the axioms. Let $p(x), q(x), r(x)$ be polynomials in \mathbb{P}_2 and let a, b, c be real numbers. Write $p(x) = p_0 + p_1x + p_2x^2$, $q(x) = q_0 + q_1x + q_2x^2$, and $r(x) = r_0 + r_1x + r_2x^2$.

- We first prove that addition of polynomials in \mathbb{P}_2 is closed. For two polynomials in \mathbb{P}_2 we need to show that their sum is also a polynomial in \mathbb{P}_2 . From the definition of \mathbb{P}_2 , a polynomial is contained in \mathbb{P}_2 if it is of degree at most 2 or the zero polynomial.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \end{aligned}$$

The sum is a polynomial of degree 2 and therefore is in \mathbb{P}_2 . It follows that \mathbb{P}_2 is closed under addition.

- We need to show that addition is commutative, that is $p(x) + q(x) = q(x) + p(x)$.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \\ &= (q_2 + p_2)x^2 + (q_1 + p_1)x + (q_0 + p_0) \\ &= q_2x^2 + q_1x + q_0 + p_2x^2 + p_1x + p_0 \\ &= q(x) + p(x) \end{aligned}$$

- Next, we need to show that addition is associative. That is, that $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$.

$$\begin{aligned} (p(x) + q(x)) + r(x) &= (p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2 + r_2)x^2 + (p_1 + q_1 + r_1)x + (p_0 + q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + (q_2 + r_2)x^2 + (q_1 + r_1)x + (q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + (q_2x^2 + q_1x + q_0 + r_2x^2 + r_1x + r_0) \\ &= p(x) + (q(x) + r(x)) \end{aligned}$$

- Next, we must prove that there exists an additive identity. Let $0(x) = 0x^2 + 0x + 0$.

$$\begin{aligned}
p(x) + 0(x) &= p_2x^2 + p_1x + p_0 + 0x^2 + 0x + 0 \\
&= (p_2 + 0)x^2 + (p_1 + 0)x + (p_0 + 0) \\
&= p_2x^2 + p_1x + p_0 \\
&= p(x)
\end{aligned}$$

Hence an additive identity exists, specifically the zero polynomial.

- Next we must prove that there exists an additive inverse. Let $-p(x) = -p_2x^2 - p_1x - p_0$ and consider the following:

$$\begin{aligned}
p(x) + (-p(x)) &= p_2x^2 + p_1x + p_0 + (-p_2x^2 - p_1x - p_0) \\
&= (p_2 - p_2)x^2 + (p_1 - p_1)x + (p_0 - p_0) \\
&= 0x^2 + 0x + 0 \\
&= 0(x)
\end{aligned}$$

Hence an additive inverse $-p(x)$ exists such that $p(x) + (-p(x)) = 0(x)$.

We now need to verify the axioms related to scalar multiplication.

- First we prove that \mathbb{P}_2 is closed under scalar multiplication. That is, we show that $ap(x)$ is also a polynomial of degree at most 2.

$$ap(x) = a(p_2x^2 + p_1x + p_0) = ap_2x^2 + ap_1x + ap_0$$

Therefore \mathbb{P}_2 is closed under scalar multiplication.

- We need to show that $a(p(x) + q(x)) = ap(x) + aq(x)$.

$$\begin{aligned}
a(p(x) + q(x)) &= a(p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0) \\
&= a((p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0)) \\
&= a(p_2 + q_2)x^2 + a(p_1 + q_1)x + a(p_0 + q_0) \\
&= (ap_2 + aq_2)x^2 + (ap_1 + aq_1)x + (ap_0 + aq_0) \\
&= ap_2x^2 + ap_1x + ap_0 + aq_2x^2 + aq_1x + aq_0 \\
&= ap(x) + aq(x)
\end{aligned}$$

- Next we show that $(a + b)p(x) = ap(x) + bp(x)$.

$$\begin{aligned}
(a + b)p(x) &= (a + b)(p_2x^2 + p_1x + p_0) \\
&= (a + b)p_2x^2 + (a + b)p_1x + (a + b)p_0 \\
&= ap_2x^2 + ap_1x + ap_0 + bp_2x^2 + bp_1x + bp_0 \\
&= ap(x) + bp(x)
\end{aligned}$$

- The next axiom which needs to be verified is $a(bp(x)) = (ab)p(x)$.

$$\begin{aligned}
 a(bp(x)) &= a(b(p_2x^2 + p_1x + p_0)) \\
 &= a(bp_2x^2 + bp_1x + bp_0) \\
 &= abp_2x^2 + abp_1x + abp_0 \\
 &= (ab)(p_2x^2 + p_1x + p_0) \\
 &= (ab)p(x)
 \end{aligned}$$

- Finally, we show that $1p(x) = p(x)$.

$$\begin{aligned}
 1p(x) &= 1(p_2x^2 + p_1x + p_0) \\
 &= 1p_2x^2 + 1p_1x + 1p_0 \\
 &= p_2x^2 + p_1x + p_0 \\
 &= p(x)
 \end{aligned}$$

Since the above axioms hold, we know that \mathbb{P}_2 as described above is a vector space. \square

Another important example of a vector space is the set of all matrices of the same size.

Example 9.4: Vector Space of Matrices

Let $\mathbb{M}_{2,3}$ be the set of all 2×3 matrices. Using the usual operations of matrix addition and scalar multiplication, show that $\mathbb{M}_{2,3}$ is a vector space.

Solution. Let A, B be 2×3 matrices in $\mathbb{M}_{2,3}$. We first prove the axioms for addition.

- In order to prove that $\mathbb{M}_{2,3}$ is closed under matrix addition, we show that the sum $A + B$ is in $\mathbb{M}_{2,3}$. This means showing that $A + B$ is a 2×3 matrix.

$$\begin{aligned}
 A + B &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}
 \end{aligned}$$

You can see that the sum is a 2×3 matrix, so it is in $\mathbb{M}_{2,3}$. It follows that $\mathbb{M}_{2,3}$ is closed under matrix addition.

- The remaining axioms regarding matrix addition follow from properties of matrix addition, found in Proposition 2.7. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of matrix addition.

We now turn our attention to the axioms regarding scalar multiplication. Let A, B be matrices in $\mathbb{M}_{2,3}$ and let c be a real number.

- We first show that $\mathbb{M}_{2,3}$ is closed under scalar multiplication. That is, we show that cA a 2×3 matrix.

$$\begin{aligned} cA &= c \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix} \end{aligned}$$

This is a 2×3 matrix in $\mathbb{M}_{2,3}$ which proves that the set is closed under scalar multiplication.

- The remaining axioms of scalar multiplication follow from properties of scalar multiplication of matrices, from Proposition 2.10. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of scalar multiplication.

In conclusion, $\mathbb{M}_{2,3}$ satisfies the required axioms and is a vector space. \square

While here we proved that the set of all 2×3 matrices is a vector space, there is nothing special about this choice of matrix size. In fact if we instead consider $\mathbb{M}_{m,n}$, the set of all $m \times n$ matrices, then $\mathbb{M}_{m,n}$ is a vector space under the operations of matrix addition and scalar multiplication.

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore *not* a vector space.

Example 9.5: Not a Vector Space

Let V denote the set of 2×3 matrices. Let addition in V be defined by $A + B = A$ for matrices A, B in V . Let scalar multiplication in V be the usual scalar multiplication of matrices. Show that V is not a vector space.

Solution. In order to show that V is not a vector space, it suffices to find only one axiom which is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. Let A, B be matrices in V .

- We first want to check if addition is closed. Consider $A + B$. By the definition of addition in the example, we have that $A + B = A$. Since A is a 2×3 matrix, it follows that the sum $A + B$ is in V , and V is closed under addition.
- We now wish to check if addition is commutative. That is, we want to check if $A + B = B + A$ **for all** choices of A and B in V . From the definition of addition, we have that $A + B = A$ and $B + A = B$. Therefore, we can find A, B in V such that these sums are not equal. One example is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using the operation defined by $A + B = A$, we have

$$\begin{aligned} A + B &= A \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B + A &= B \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

It follows that $A + B \neq B + A$. Therefore addition as defined for V is not commutative and V fails this axiom. Hence V is not a vector space.

□

Consider another example of a vector space.

Example 9.6: Vector Space of Functions

Let S be a nonempty set and define \mathbb{F}_S to be the set of real functions defined on S . In other words, we write $\mathbb{F}_S : S \mapsto \mathbb{R}$. Letting a, b, c be scalars and f, g, h functions, the vector operations are defined as

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (af)(x) &= a(f(x)) \end{aligned}$$

Show that \mathbb{F}_S is a vector space.

Solution. To verify that \mathbb{F}_S is a vector space, we must prove the axioms beginning with those for addition. Let f, g, h be functions in \mathbb{F}_S .

- First we check that addition is closed. For functions f, g defined on the set S , their sum given by

$$(f + g)(x) = f(x) + g(x)$$

is again a function defined on S . Hence this sum is in \mathbb{F}_S and \mathbb{F}_S is closed under addition.

- Secondly, we check the commutative law of addition:

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$$

Since x is arbitrary, $f + g = g + f$.

- Next we check the associative law of addition:

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = (f(x) + (g + h)(x)) = (f + (g + h))(x) \end{aligned}$$

and so $(f + g) + h = f + (g + h)$.

- Next we check for an additive identity. Let 0 denote the function which is given by $0(x) = 0$. Then this is an additive identity because

$$(f + 0)(x) = f(x) + 0(x) = f(x)$$

and so $f + 0 = f$.

- Finally, check for an additive inverse. Let $-f$ be the function which satisfies $(-f)(x) = -f(x)$. Then

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0$$

Hence $f + (-f) = 0$.

Now, check the axioms for scalar multiplication.

- We first need to check that \mathbb{F}_S is closed under scalar multiplication. For a function $f(x)$ in \mathbb{F}_S and real number a , the function $(af)(x) = a(f(x))$ is again a function defined on the set S . Hence $a(f(x))$ is in \mathbb{F}_S and \mathbb{F}_S is closed under scalar multiplication.

•

$$((a + b)f)(x) = (a + b)f(x) = af(x) + bf(x) = (af + bf)(x)$$

and so $(a + b)f = af + bf$.

•

$$\begin{aligned} (a(f + g))(x) &= a(f + g)(x) = a(f(x) + g(x)) \\ &= af(x) + ag(x) = (af + ag)(x) \end{aligned}$$

and so $a(f + g) = af + ag$.

•

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = (a(bf))(x)$$

so $(abf) = a(bf)$.

- Finally $(1f)(x) = 1f(x) = f(x)$ so $1f = f$.

It follows that V satisfies all the required axioms and is a vector space. \square

We conclude this section with the following important theorem.

Theorem 9.7: Uniqueness

In any vector space, the following are true:

1. $\vec{0}$, the additive identity, is unique
2. $-\vec{x}$, the additive inverse, is unique
3. $0\vec{x} = \vec{0}$ for all vectors \vec{x}
4. $(-1)\vec{x} = -\vec{x}$ for all vectors \vec{x}

Proof.

1. When we say that the additive identity, $\vec{0}$, is unique, we mean that if a vector acts like the additive identity, then it **is** the additive identity. To prove this uniqueness, we want to show that another vector which acts like the additive identity is actually equal to $\vec{0}$.

Suppose $\vec{0}'$ is also an additive identity. Then,

$$\vec{0} + \vec{0}' = \vec{0}$$

Now, for $\vec{0}$ the additive identity given above in the axioms, we have that

$$\vec{0}' + \vec{0} = \vec{0}'$$

So by commutativity:

$$0 = 0 + 0' = 0' + 0 = 0'$$

This says that if a vector acts like an additive identity (such as $\vec{0}'$), it in fact equals $\vec{0}$. This proves the uniqueness of $\vec{0}$.

2. When we say that the additive inverse, $-\vec{x}$, is unique, we mean that if a vector acts like the additive inverse, then it **is** the additive inverse. Suppose that \vec{y} acts like an additive inverse:

$$\vec{x} + \vec{y} = \vec{0}$$

Then the following holds:

$$\vec{y} = \vec{0} + \vec{y} = (-\vec{x} + \vec{x}) + \vec{y} = -\vec{x} + (\vec{x} + \vec{y}) = -\vec{x} + \vec{0} = -\vec{x}$$

Thus if \vec{y} acts like the additive inverse, it is equal to the additive inverse $-\vec{x}$. This proves the uniqueness of $-\vec{x}$.

3. This statement claims that for all vectors \vec{x} , scalar multiplication by 0 equals the zero vector $\vec{0}$. Consider the following, using the fact that we can write $0 = 0 + 0$:

$$0\vec{x} = (0 + 0)\vec{x} = 0\vec{x} + 0\vec{x}$$

We use a small trick here: add $-0\vec{x}$ to both sides. This gives

$$\begin{aligned} 0\vec{x} + (-0\vec{x}) &= 0\vec{x} + 0\vec{x} + (-\vec{x}) \\ \vec{0} + 0 &= 0\vec{x} + 0 \\ \vec{0} &= 0\vec{x} \end{aligned}$$

This proves that scalar multiplication of any vector by 0 results in the zero vector $\vec{0}$.

4. Finally, we wish to show that scalar multiplication of -1 and any vector \vec{x} results in the additive inverse of that vector, $-\vec{x}$. Recall from 2. above that the additive inverse is unique. Consider the following:

$$\begin{aligned} (-1)\vec{x} + \vec{x} &= (-1)\vec{x} + 1\vec{x} \\ &= (-1 + 1)\vec{x} \\ &= 0\vec{x} \\ &= \vec{0} \end{aligned}$$

By the uniqueness of the additive inverse shown earlier, any vector which acts like the additive inverse must be equal to the additive inverse. It follows that $(-1)\vec{x} = -\vec{x}$.

□

9.1.1. EXERCISES

1. Let X consist of the real valued functions which are defined on an interval $[a, b]$. For $f, g \in X$, $f + g$ is the name of the function which satisfies $(f + g)(x) = f(x) + g(x)$. For s a real number, $(sf)(x) = s(f(x))$. Show this is a vector space.
2. Consider functions defined on $\{1, 2, \dots, n\}$ having values in \mathbb{R} . Explain how, if V is the set of all such functions, V can be considered as \mathbb{R}^n .
3. Let the vectors be polynomials of degree no more than 3. Show that with the usual definitions of scalar multiplication and addition wherein, for $p(x)$ a polynomial, $(ap)(x) = ap(x)$ and for p, q polynomials $(p + q)(x) = p(x) + q(x)$, this is a vector space.

9.2 SUBSPACES

Outcomes

- A. Determine if a vector is within a given span.
- B. Utilize the subspace test to determine if a set is a subspace of a given vector space.

In this section we will examine the concepts of spanning and subspaces introduced earlier in terms of \mathbb{R}^n . Here, we will discuss these concepts in terms of abstract vector spaces. Consider the following definition.

Definition 9.8: Subset

Let X and Y be two sets. If all elements of X are also elements of Y then we say that X is a **subset** of Y and we write

$$X \subseteq Y$$

In particular, we often speak of subsets of a vector space, such as $X \subseteq V$. By this we mean that every element in the set X is contained in the vector space V .

Definition 9.9: Span of Vectors

Let $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$. Then

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \left\{ \sum_{i=1}^n c_i \vec{v}_i : c_i \in \mathbb{R} \right\}$$

When we say that a vector \vec{w} is in $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ we mean that \vec{w} can be written as a linear combination of the \vec{v}_i .

Consider the following example.

Example 9.10: Matrix Span

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if A and B are in

$$\text{span}\{M_1, M_2\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution.

First consider A . We want to see if scalars s, t can be found such that $A = sM_1 + tM_2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by

$$\begin{aligned} 1 &= s \\ 2 &= t \end{aligned}$$

and it follows that A is in $\text{span}\{M_1, M_2\}$.

Now consider B . Again we write $B = sM_1 + tM_2$ and see if a solution can be found for s, t .

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly no values of s and t can be found such that this equation holds. Therefore B is not in $\text{span}\{M_1, M_2\}$. \square

Consider another example.

Example 9.11: Polynomial Span

Show that $p(x) = 7x^2 + 4x - 3$ is in $\text{span}\{4x^2 + x, x^2 - 2x + 3\}$.

Solution. To show that $p(x)$ is in the given span, we need to show that it can be written as a linear combination of polynomials in the span. Suppose scalars a, b existed such that

$$7x^2 + 4x - 3 = a(4x^2 + x) + b(x^2 - 2x + 3)$$

If this linear combination were to hold, the following would be true:

$$\begin{aligned} 4a + b &= 7 \\ a - 2b &= 4 \\ 3b &= -3 \end{aligned}$$

You can verify that $a = 2, b = -1$ satisfies this system of equations. This means that we can write $p(x)$ as follows:

$$7x^2 + 4x - 3 = 2(4x^2 + x) - (x^2 - 2x + 3)$$

Hence $p(x)$ is in the given span. □

Consider the following example.

Example 9.12: Spanning Set

Let $S = \{x^2 + 1, x - 2, 2x^2 - x\}$. Show that S is a spanning set for \mathbb{P}_2 , the set of all polynomials of degree at most 2.

Solution. Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in \mathbb{P}_2 . To show that S is a spanning set, it suffices to show that $p(x)$ can be written as a linear combination of the elements of S . In other words, can we find r, s, t such that:

$$p(x) = ax^2 + bx + c = r(x^2 + 1) + s(x - 2) + t(2x^2 - x)$$

If a solution r, s, t can be found, then this shows that for any such polynomial $p(x)$, it can be written as a linear combination of the above polynomials and S is a spanning set.

$$\begin{aligned} ax^2 + bx + c &= r(x^2 + 1) + s(x - 2) + t(2x^2 - x) \\ &= rx^2 + r + sx - 2s + 2tx^2 - tx \\ &= (r + 2t)x^2 + (s - t)x + (r - 2s) \end{aligned}$$

For this to be true, the following must hold:

$$\begin{aligned} a &= r + 2t \\ b &= s - t \\ c &= r - 2s \end{aligned}$$

To check that a solution exists, set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a + 2b + \frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a - \frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a - b - \frac{1}{4}c \end{array} \right]$$

Clearly a solution exists for any choice of a, b, c . Hence S is a spanning set for \mathbb{P}_2 . \square

Consider now the definition of a subspace.

Definition 9.13: Subspace

Let V be a vector space. A subset $W \subseteq V$ is said to be a **subspace** of V if $a\vec{x} + b\vec{y} \in W$ whenever $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in W$.

The span of a set of vectors as described in Definition 9.9 is an example of a subspace. The following fundamental result says that subspaces are subsets of a vector space which are themselves vector spaces.

Theorem 9.14: Subspaces are Vector Spaces

Let W be a nonempty collection of vectors in V , a vector space. Then W is a subspace if and only if W is itself a vector space having the same operations as those defined on V .

Proof. Suppose first that W is a subspace. It is obvious that all the algebraic laws hold on W because it is a subset of V and they hold on V . Thus $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ along with the other axioms. Does W contain $\vec{0}$? Yes because it contains $0\vec{u} = \vec{0}$. See Theorem 9.7.

Are the operations of V defined on W ? That is, when you add vectors of W do you get a vector in W ? When you multiply a vector in W by a scalar, do you get a vector in W ? Yes. This is contained in the definition. Does every vector in W have an additive inverse? Yes by Theorem 9.7 because $-\vec{v} = (-1)\vec{v}$ which is given to be in W provided $\vec{v} \in W$.

Next suppose W is a vector space. Then by definition, it is closed with respect to linear combinations. Hence it is a subspace. \square

When determining spanning sets the following theorem proves useful.

Theorem 9.15: Spanning Set

Let $S \subseteq V$ for a vector space V and suppose $S = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Let $R \subseteq V$ be a subspace such that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in R$. Then it follows that $S \subseteq R$.

In other words, this theorem claims that any subspace that contains a set of vectors must also contain the span of these vectors.

The following example will show that two spans, described differently, can in fact be equal.

Example 9.16: Equal Span

Let $p(x), q(x)$ be polynomials and suppose $U = \text{span}\{2p(x) - q(x), p(x) + 3q(x)\}$ and $W = \text{span}\{p(x), q(x)\}$. Show that $U = W$.

Solution. We will use Theorem 9.15 to show that $U \subseteq W$ and $W \subseteq U$. It will then follow that $U = W$.

1. $U \subseteq W$

Notice that $2p(x) - q(x)$ and $p(x) + 3q(x)$ are both in $W = \text{span}\{p(x), q(x)\}$. Then by Theorem 9.15 W must contain the span of these polynomials and so $U \subseteq W$.

2. $W \subseteq U$

Notice that

$$\begin{aligned} p(x) &= \frac{3}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \\ q(x) &= -\frac{1}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \end{aligned}$$

Hence $p(x), q(x)$ are in $\text{span}\{2p(x) - q(x), p(x) + 3q(x)\}$. By Theorem 9.15 U must contain the span of these polynomials and so $W \subseteq U$.

□

To prove that a set is a vector space, one must verify each of the axioms given in Definition 9.1. This is a cumbersome task, and therefore a shorter procedure is used to verify a subspace.

Procedure 9.17: Subspace Test

Suppose W is a subset of a vector space V . The W is a **subspace** of W if the following three conditions hold, using the operations of V :

1. The additive identity $\vec{0}$ of V is contained in W .
2. For any vectors \vec{w}_1, \vec{w}_2 in W , $\vec{w}_1 + \vec{w}_2$ is also in W .
3. For any vector \vec{w}_1 in W and scalar a , the product $a\vec{w}_1$ is also in W .

Therefore it suffices to prove these three steps to show that a set is a subspace. Consider the following example.

Example 9.18: Improper Subspaces

Let V be an arbitrary vector space. Then V is a subspace of itself. Similarly, the set $\{\vec{0}\}$ containing only the zero vector is also a subspace.

Solution. Using the subspace test in Procedure 9.17 we can show that V and $\{\vec{0}\}$ are subspaces of V .

The conditions are clearly satisfied by V , since V satisfies the vector space axioms. Therefore V is a subspace.

Let's consider the set $\{\vec{0}\}$.

1. The vector $\vec{0}$ is clearly contained in $\{\vec{0}\}$, so the first condition is satisfied.

2. Let \vec{w}_1, \vec{w}_2 be in $\{\vec{0}\}$. Then $\vec{w}_1 = \vec{0}$ and $\vec{w}_2 = \vec{0}$ and so

$$\vec{w}_1 + \vec{w}_2 = \vec{0} + \vec{0} = \vec{0}$$

It follows that the sum is contained in $\{\vec{0}\}$ and the second condition is satisfied.

3. Let \vec{w}_1 be in $\{\vec{0}\}$ and let a be an arbitrary scalar. Then

$$a\vec{w}_1 = a\vec{0} = \vec{0}$$

Hence the product is contained in $\{\vec{0}\}$ and the third condition is satisfied.

It follows that $\{\vec{0}\}$ is a subspace of V . □

The two subspaces described above are called **improper subspaces**. Any subspace of a vector space V which is not equal to V or $\{\vec{0}\}$ is called a **proper subspace**.

Consider another example.

Example 9.19: Subspace of Polynomials

Let \mathbb{P}_2 be the vector space of polynomials of degree two or less. Let $W \subseteq \mathbb{P}_3$ be all polynomials of degree two or less which have 1 as a root. Show that W is a subspace of \mathbb{P}_3 .

Solution. First, express W as follows:

$$W = \{p(x) = ax^2 + bx + c, a, b, c, \in \mathbb{R} | p(1) = 0\}$$

We need to show that W satisfies the three conditions of Procedure 9.17.

1. The zero polynomial of \mathbb{P}_2 is given by $0(x) = 0x^2 + 0x + 0 = 0$. Clearly $0(1) = 0$ so $0(x)$ is contained in W .
2. Let $p(x), q(x)$ be polynomials in W . It follows that $p(1) = 0$ and $q(1) = 0$. Now consider $p(x) + q(x)$. Let $r(x)$ represent this sum.

$$\begin{aligned} r(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore the sum is also in W and the second condition is satisfied.

3. Let $p(x)$ be a polynomial in W and let a be a scalar. It follows that $p(1) = 0$. Consider the product $ap(x)$.

$$\begin{aligned} ap(1) &= a(0) \\ &= 0 \end{aligned}$$

Therefore the product is in W and the third condition is satisfied.

It follows that W is a subspace of \mathbb{P}_2 . □

9.2.1. EXERCISES

1. Let V be a vector space and suppose $\{\vec{x}_1, \dots, \vec{x}_k\}$ is a set of vectors in V . Show that $\vec{0}$ is in $\text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$.

2. Determine if $p(x) = 4x^2 - x$ is in the span given by

$$\text{span}\{x^2 + x, x^2 - 1, -x + 2\}$$

3. Determine if $p(x) = -x^2 + x + 2$ is in the span given by

$$\text{span}\{x^2 + x + 1, 2x^2 + x\}$$

4. Determine if $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ is in the span given by

$$\text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

5. Show that the spanning set in Question 4 is a spanning set for M_{22} , the vector space of all 2×2 matrices.

6. Let $M = \{\vec{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : |u_1| \leq 4\}$. Is M a subspace of \mathbb{R}^4 ?

7. Let $M = \{\vec{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \sin(u_1) = 1\}$. Is M a subspace of \mathbb{R}^4 ?

8. Let W be a subset of M_{22} given by

$$W = \{A | A \in M_{22}, A^T = A\}$$

In words, W is the set of all symmetric 2×2 matrices. Is W a subspace of M_{22} ?

9. Let W be a subset of M_{22} given by

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, a + b = c + d \right\}$$

Is W a subspace of M_{22} ?

10. Let W be a subset of P_3 given by

$$W = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}, d = 0\}$$

Is W a subspace of P_3 ?

11. Let W be a subset of P_3 given by

$$W = \{p(x) = ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}, p(2) = 1\}$$

Is W a subspace of P_3 ?

9.3 LINEAR INDEPENDENCE AND BASES

Outcomes

- A. Determine if a set is linearly independent.
- B. Extend a linearly independent set and shrink a spanning set to a basis of a given vector space.

In this section, we will again explore concepts introduced earlier in terms of \mathbb{R}^n and extend them to apply to abstract vector spaces.

Definition 9.20: Linear Independence

Let V be a vector space. If $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$, then it is linearly independent if

$$\sum_{i=1}^n a_i \vec{v}_i = \vec{0} \text{ implies } a_1 = \dots = a_n = 0$$

where the a_i are real numbers.

The set of vectors is called linearly dependent if it is not linearly independent.

Example 9.21: Linear Independence

Let $S \subseteq \mathbb{P}_2$ be a set of polynomials given by

$$S = \{x^2 + 2x - 1, 2x^2 - x + 3\}$$

Determine if S is linearly independent.

Solution. To determine if this set S is linearly independent, we write

$$a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0x^2 + 0x + 0$$

If it is linearly independent, then $a = b = 0$ will be the only solution. We proceed as follows.

$$\begin{aligned} a(x^2 + 2x - 1) + b(2x^2 - x + 3) &= 0x^2 + 0x + 0 \\ ax^2 + 2ax - a + 2bx^2 - bx + 3b &= 0x^2 + 0x + 0 \\ (a + 2b)x^2 + (2a - b)x - a + 3b &= 0x^2 + 0x + 0 \end{aligned}$$

It follows that

$$\begin{aligned} a + 2b &= 0 \\ 2a - b &= 0 \\ -a + 3b &= 0 \end{aligned}$$

The augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Hence the solution is $a = b = 0$ and the set is linearly independent. \square

If we know that one particular set is linearly independent, we can use this information to determine if a related set is linearly independent. Consider the following example.

Example 9.22: Related Independent Sets

Let V be a vector space and suppose $S \subseteq V$ is a set of linearly independent vectors given by $S = \{\vec{u}, \vec{v}, \vec{w}\}$. Let $R \subseteq V$ be given by $R = \{2\vec{u} - \vec{w}, \vec{w} + \vec{v}, 3\vec{v} + \frac{1}{2}\vec{u}\}$. Show that R is also linearly independent.

Solution. To determine if R is linearly independent, we write

$$a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) = \vec{0}$$

If the set is linearly independent, the only solution will be $a = b = c = 0$. We proceed as follows.

$$\begin{aligned} a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) &= \vec{0} \\ 2a\vec{u} - a\vec{w} + b\vec{w} + b\vec{v} + 3c\vec{v} + \frac{1}{2}c\vec{u} &= \vec{0} \\ (2a + \frac{1}{2}c)\vec{u} + (b + 3c)\vec{v} + (-a + b)\vec{w} &= \vec{0} \end{aligned}$$

We know that the set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, which implies that the coefficients in the last line of this equation must all equal 0. In other words:

$$\begin{aligned} 2a + \frac{1}{2}c &= 0 \\ b + 3c &= 0 \\ -a + b &= 0 \end{aligned}$$

The augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Hence the solution is $a = b = c = 0$ and the set is linearly independent. \square

Consider the span of a linearly independent set of vectors. Suppose we take a vector which is not in this span and add it to the set. The following lemma claims that the resulting set is still linearly independent.

Lemma 9.23: Adding to a Linearly Independent Set

Suppose $\vec{v} \notin \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$ and $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent. Then the set

$$\{\vec{u}_1, \dots, \vec{u}_k, \vec{v}\}$$

is also linearly independent.

Proof. Suppose $\sum_{i=1}^k c_i \vec{u}_i + d\vec{v} = \vec{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for \vec{v} as a linear combination of the vectors, $\{\vec{u}_1, \dots, \vec{u}_k\}$,

$$\vec{v} = -\sum_{i=1}^k \left(\frac{c_i}{d}\right) \vec{u}_i$$

contrary to the assumption that \vec{v} is not in the span of the \vec{u}_i . Therefore, $d = 0$. But then $\sum_{i=1}^k c_i \vec{u}_i = \vec{0}$ and the linear independence of $\{\vec{u}_1, \dots, \vec{u}_k\}$ implies each $c_i = 0$ also. \square

Consider the following example.

Example 9.24: Adding to a Linearly Independent Set

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Show that the set $R \subseteq M_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent.

Solution. Instead of writing a linear combination of the matrices which equals 0 and showing that the coefficients must equal 0, we can instead use Lemma 9.23.

To do so, we show that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Write

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Clearly there are no possible a, b to make this equation true. Hence the new matrix does not lie in the span of the matrices in S . By Lemma 9.23, R is also linearly independent. \square

Recall the definition of basis, considered now in the context of vector spaces.

Definition 9.25: Basis

Let V be a vector space. Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is called a basis for V if the following conditions hold.

1. $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$
2. $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent

Consider the following example.

Example 9.26: Polynomials of Degree Two

Let \mathbb{P}_2 be the set polynomials of degree no more than 2. We can write $\mathbb{P}_2 = \text{span}\{x^2, x, 1\}$. Is $\{x^2, x, 1\}$ a basis for \mathbb{P}_2 ?

Solution. It can be verified that \mathbb{P}_2 is a vector space defined under the usual addition and scalar multiplication of polynomials.

Now, since $\mathbb{P}_2 = \text{span}\{x^2, x, 1\}$, the set $\{x^2, x, 1\}$ is a basis if it is linearly independent. Suppose then that

$$ax^2 + bx + c = 0x^2 + 0x + 0$$

where a, b, c are real numbers. It is clear that this can only occur if $a = b = c = 0$. Hence the set is linearly independent and forms a basis of \mathbb{P}_2 . \square

The next theorem is an essential result in linear algebra and is called the exchange theorem.

Theorem 9.27: Exchange Theorem

Let $\{\vec{x}_1, \dots, \vec{x}_r\}$ be a linearly independent set of vectors such that each \vec{x}_i is contained in $\text{span}\{\vec{y}_1, \dots, \vec{y}_s\}$. Then $r \leq s$.

Proof. The proof will proceed as follows. First, we set up the necessary steps for the proof. Next, we will assume that $r > s$ and show that this leads to a contradiction, thus requiring that $r \leq s$.

Define $\text{span}\{\vec{y}_1, \dots, \vec{y}_s\} = V$. Since each \vec{x}_i is in $\text{span}\{\vec{y}_1, \dots, \vec{y}_s\}$, it follows there exist scalars c_1, \dots, c_s such that

$$\vec{x}_1 = \sum_{i=1}^s c_i \vec{y}_i \quad (9.1)$$

Note that not all of these scalars c_i can equal zero. Suppose that all the $c_i = 0$. Then it would follow that $\vec{x}_1 = \vec{0}$ and so $\{\vec{x}_1, \dots, \vec{x}_r\}$ would not be linearly independent. Indeed, if $\vec{x}_1 = \vec{0}$, $1\vec{x}_1 + \sum_{i=2}^r 0\vec{x}_i = \vec{x}_1 = \vec{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\vec{x}_1, \dots, \vec{x}_r\}$ which equals zero. Therefore at least one c_i is nonzero.

Say $c_k \neq 0$. Then solve 9.1 for \vec{y}_k and obtain

$$\vec{y}_k \in \text{span} \left\{ \vec{x}_1, \overbrace{\vec{y}_1, \dots, \vec{y}_{k-1}, \vec{y}_{k+1}, \dots, \vec{y}_s}^{\text{s-1 vectors here}} \right\}.$$

Define $\{\vec{z}_1, \dots, \vec{z}_{s-1}\}$ to be

$$\{\vec{z}_1, \dots, \vec{z}_{s-1}\} = \{\vec{y}_1, \dots, \vec{y}_{k-1}, \vec{y}_{k+1}, \dots, \vec{y}_s\}$$

Now we can write

$$\vec{y}_k \in \text{span} \{\vec{x}_1, \vec{z}_1, \dots, \vec{z}_{s-1}\}$$

Therefore, $\text{span}\{\vec{x}_1, \vec{z}_1, \dots, \vec{z}_{s-1}\} = V$. To see this, suppose $\vec{v} \in V$. Then there exist constants c_1, \dots, c_s such that

$$\vec{v} = \sum_{i=1}^{s-1} c_i \vec{z}_i + c_s \vec{y}_k.$$

Replace this \vec{y}_k with a linear combination of the vectors $\{\vec{x}_1, \vec{z}_1, \dots, \vec{z}_{s-1}\}$ to obtain $\vec{v} \in \text{span}\{\vec{x}_1, \vec{z}_1, \dots, \vec{z}_{s-1}\}$. The vector \vec{y}_k , in the list $\{\vec{y}_1, \dots, \vec{y}_s\}$, has now been replaced with the vector \vec{x}_1 and the resulting modified list of vectors has the same span as the original list of vectors, $\{\vec{y}_1, \dots, \vec{y}_s\}$.

We are now ready to move on to the proof. Suppose that $r > s$ and that $\text{span}\{\vec{x}_1, \dots, \vec{x}_l, \vec{z}_1, \dots, \vec{z}_p\} = V$, where the process established above has continued. In other words, the vectors $\vec{z}_1, \dots, \vec{z}_p$ are each taken from the set $\{\vec{y}_1, \dots, \vec{y}_s\}$ and $l + p = s$. This was done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l + 1 \leq r$. Therefore, \vec{x}_{l+1} is a vector not in the list, $\{\vec{x}_1, \dots, \vec{x}_l\}$ and since $\text{span}\{\vec{x}_1, \dots, \vec{x}_l, \vec{z}_1, \dots, \vec{z}_p\} = V$ there exist scalars, c_i and d_j such that

$$\vec{x}_{l+1} = \sum_{i=1}^l c_i \vec{x}_i + \sum_{j=1}^p d_j \vec{z}_j. \quad (9.2)$$

Not all the d_j can equal zero because if this were so, it would follow that $\{\vec{x}_1, \dots, \vec{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, 9.2 can be solved for one of the \vec{z}_i , say \vec{z}_k , in terms of \vec{x}_{l+1} and the other \vec{z}_i and just as in the above argument, replace that \vec{z}_i with \vec{x}_{l+1} to obtain

$$\text{span} \left\{ \vec{x}_1, \dots, \vec{x}_l, \vec{x}_{l+1}, \overbrace{\vec{z}_1, \dots, \vec{z}_{k-1}, \vec{z}_{k+1}, \dots, \vec{z}_p}^{\text{p-1 vectors here}} \right\} = V$$

Continue this way, eventually obtaining

$$\text{span}\{\vec{x}_1, \dots, \vec{x}_s\} = V.$$

But then $\vec{x}_r \in \text{span}\{\vec{x}_1, \dots, \vec{x}_s\}$ contrary to the assumption that $\{\vec{x}_1, \dots, \vec{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed. \square

The following corollary follows from the exchange theorem.

Corollary 9.28: Two Bases of the Same Length

Let $\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_n \in V$. If $\{\vec{u}_1, \dots, \vec{u}_m\}$ and $\{\vec{v}_1, \dots, \vec{v}_n\}$ are two bases for V , then $m = n$.

Proof. By Theorem 9.27, $m \leq n$ and $n \leq m$. Therefore $m = n$. \square

This corollary is very important so we provide another proof independent of the exchange theorem above.

Proof. Suppose $n > m$. Then since the vectors $\{\vec{u}_1, \dots, \vec{u}_m\}$ span V , there exist scalars c_{ij} such that

$$\sum_{i=1}^m c_{ij} \vec{u}_i = \vec{v}_j.$$

Therefore,

$$\sum_{j=1}^n d_j \vec{v}_j = \vec{0} \text{ if and only if } \sum_{j=1}^n \sum_{i=1}^m c_{ij} d_j \vec{u}_i = \vec{0}$$

if and only if

$$\sum_{i=1}^m \left(\sum_{j=1}^n c_{ij} d_j \right) \vec{u}_i = \vec{0}$$

Now since $\{\vec{u}_1, \dots, \vec{u}_m\}$ is independent, this happens if and only if

$$\sum_{j=1}^n c_{ij} d_j = 0, \quad i = 1, 2, \dots, m.$$

However, this is a system of m equations in n variables, d_1, \dots, d_n and $m < n$. Therefore, there exists a solution to this system of equations in which not all the d_j are equal to zero. Recall why this is so. The augmented matrix for the system is of the form $\left[C \mid \vec{0} \right]$ where C is a matrix which has more columns than rows. Therefore, there are free variables and hence nonzero solutions to the system of equations. However, this contradicts the linear independence of $\{\vec{u}_1, \dots, \vec{u}_m\}$. Similarly it cannot happen that $m > n$. \square

Given the result of the previous corollary, the following definition follows.

Definition 9.29: Dimension

A vector space V is of dimension n if it has a basis consisting of n vectors.

Notice that the dimension is well defined by Corollary 9.28. It is assumed here that $n < \infty$ and therefore such a vector space is said to be finite dimensional.

Example 9.30: Dimension of a Vector Space

Let \mathbb{P}_2 be the set of all polynomials of degree at most 2. Find the dimension of \mathbb{P}_2 .

Solution. If we can find a basis of \mathbb{P}_2 then the number of vectors in the basis will give the dimension. Recall from Example 9.26 that a basis of \mathbb{P}_2 is given by

$$S = \{x^2, x, 1\}$$

There are three polynomials in S and hence the dimension of \mathbb{P}_2 is three. \square

It is important to note that a basis for a vector space is not unique. A vector space can have many bases. Consider the following example.

Example 9.31: A Different Basis for Polynomials of Degree Two

Let \mathbb{P}_2 be the polynomials of degree no more than 2. Is $\{x^2 + x + 1, 2x + 1, 3x^2 + 1\}$ a basis for \mathbb{P}_2 ?

Solution. Suppose these vectors are linearly independent but do not form a spanning set for \mathbb{P}_2 . Then by Lemma 9.23, we could find a fourth polynomial in \mathbb{P}_2 to create a new linearly independent set containing four polynomials. However this would imply that we could find a basis of \mathbb{P}_2 of more than three polynomials. This contradicts the result of Example 9.30 in which we determined the dimension of \mathbb{P}_2 is three. Therefore if these vectors are linearly independent they must also form a spanning set and thus a basis for \mathbb{P}_2 .

Suppose then that

$$\begin{aligned}a(x^2 + x + 1) + b(2x + 1) + c(3x^2 + 1) &= 0 \\(a + 3c)x^2 + (a + 2b)x + (a + b + c) &= 0\end{aligned}$$

We know that $\{1, x, x^2\}$ is linearly independent, and so it follows that

$$\begin{aligned}a + 3c &= 0 \\a + 2b &= 0 \\a + b + c &= 0\end{aligned}$$

and there is only one solution to this system of equations, $a = b = c = 0$. Therefore, these are linearly independent and form a basis for \mathbb{P}_2 . \square

Consider the following theorem.

Theorem 9.32: Every Subspace has a Basis

Let V be a nonzero subspace of a finite dimensional vector space W . Suppose W has dimension n . Then V has a basis with no more than n vectors.

Proof. Let $\vec{v}_1 \in V$ where $\vec{v}_1 \neq 0$. If $\text{span}\{\vec{v}_1\} = V$, then it follows that $\{\vec{v}_1\}$ is a basis for V . Otherwise, there exists $\vec{v}_2 \in V$ which is not in $\text{span}\{\vec{v}_1\}$. By Lemma 9.23 $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of vectors. Then $\{\vec{v}_1, \vec{v}_2\}$ is a basis for V and we are done. If $\text{span}\{\vec{v}_1, \vec{v}_2\} \neq V$, then there exists $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n + 1$ steps because if not, it would be possible to obtain $n + 1$ linearly independent vectors contrary to the exchange theorem, Theorem 9.27. \square

The following theorem claims that a spanning set of a vector space V can be shrunk down to a basis of V . Similarly, a linearly independent set within V can be enlarged to create a basis of V .

Theorem 9.33: Basis of V

If $V = \text{span}\{\vec{u}_1, \dots, \vec{u}_n\}$ is a vector space, then some subset of $\{\vec{u}_1, \dots, \vec{u}_n\}$ is a basis for V . Also, if $\{\vec{u}_1, \dots, \vec{u}_k\} \subseteq V$ is linearly independent and the vector space is finite dimensional, then the set $\{\vec{u}_1, \dots, \vec{u}_k\}$, can be enlarged to obtain a basis of V .

Proof. Let

$$S = \{E \subseteq \{\vec{u}_1, \dots, \vec{u}_n\} \text{ such that } \text{span}\{E\} = V\}.$$

For $E \in S$, let $|E|$ denote the number of elements of E . Let

$$m = \min\{|E| \text{ such that } E \in S\}.$$

Thus there exist vectors

$$\{\vec{v}_1, \dots, \vec{v}_m\} \subseteq \{\vec{u}_1, \dots, \vec{u}_n\}$$

such that

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_m\} = V$$

and m is as small as possible for this to happen. If this set is linearly independent, it follows it is a basis for V and the theorem is proved. On the other hand, if the set is not linearly independent, then there exist scalars, c_1, \dots, c_m such that

$$\vec{0} = \sum_{i=1}^m c_i \vec{v}_i$$

and not all the c_i are equal to zero. Suppose $c_k \neq 0$. Then solve for the vector \vec{v}_k in terms of the other vectors. Consequently,

$$V = \text{span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_{k+1}, \dots, \vec{v}_m\}$$

contradicting the definition of m . This proves the first part of the theorem.

To obtain the second part, begin with $\{\vec{u}_1, \dots, \vec{u}_k\}$ and suppose a basis for V is

$$\{\vec{v}_1, \dots, \vec{v}_n\}$$

If

$$\text{span}\{\vec{u}_1, \dots, \vec{u}_k\} = V,$$

then $k = n$. If not, there exists a vector

$$\vec{u}_{k+1} \notin \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$$

Then from Lemma 9.23, $\{\vec{u}_1, \dots, \vec{u}_k, \vec{u}_{k+1}\}$ is also linearly independent. Continue adding vectors in this way until n linearly independent vectors have been obtained. Then

$$\text{span}\{\vec{u}_1, \dots, \vec{u}_n\} = V$$

because if it did not do so, there would exist \vec{u}_{n+1} as just described and $\{\vec{u}_1, \dots, \vec{u}_{n+1}\}$ would be a linearly independent set of vectors having $n+1$ elements. This contradicts the fact that

$\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis. In turn this would contradict Theorem 9.27. Therefore, this list is a basis. \square

Consider the following example.

Example 9.34: Shrinking a Spanning Set

Consider the set $S \subseteq \mathbb{P}_2$ given by

$$S = \{1, x, x^2, x^2 + 1\}$$

Show that S spans \mathbb{P}_2 , then remove vectors from S until it creates a basis.

Solution. First we need to show that S spans \mathbb{P}_2 . Let $ax^2 + bx + c$ be an arbitrary polynomial in \mathbb{P}_2 . Write

$$ax^2 + bx + c = r(1) + s(x) + t(x^2) + u(x^2 + 1)$$

Then,

$$\begin{aligned} ax^2 + bx + c &= r(1) + s(x) + t(x^2) + u(x^2 + 1) \\ &= (t + u)x^2 + s(x) + (r + u) \end{aligned}$$

It follows that

$$\begin{aligned} a &= t + u \\ b &= s \\ c &= r + u \end{aligned}$$

Clearly a solution exists for all a, b, c and so S is a spanning set for \mathbb{P}_2 . By Theorem 9.33, some subset of S is a basis for \mathbb{P}_2 .

Recall that a basis must be both a spanning set and a linearly independent set. Therefore we must remove a vector from S keeping this in mind. Suppose we remove x from S . The resulting set would be $\{1, x^2, x^2 + 1\}$. This set is clearly linearly dependent (and also does not span \mathbb{P}_2) and so is not a basis.

Suppose we remove $x^2 + 1$ from S . The resulting set is $\{1, x, x^2\}$ which is both linearly independent and spans \mathbb{P}_2 . Hence this is a basis for \mathbb{P}_2 . Note that removing $1, x^2$, or $x^2 + 1$ will result in a basis. \square

Recall Example 9.24 in which we added a matrix to a linearly independent set to create a larger linearly independent set. By Theorem 9.33 we can extend a linearly independent set to a basis.

Example 9.35: Adding to a Linearly Independent Set

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Enlarge S to a basis of M_{22} .

Solution. Recall from the solution of Example 9.24 that the set $R \subseteq M_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent. However this set is still not a basis for M_{22} as it is not a spanning set. In particular, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in $\text{span}R$. Therefore, this matrix can be added to the set by Lemma 9.23 to obtain a new linearly independent set given by

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This set is linearly independent and now spans M_{22} . Hence T is a basis. \square

9.3.1. EXERCISES

1. Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\{x + 1, x^2 + 2, x^2 - x - 3\}$$

2. Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\{x^2 + x, -2x^2 - 4x - 6, 2x - 2\}$$

3. Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -7 & 2 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \right\}$$

4. Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

5. If you have 5 vectors in \mathbb{R}^5 and the vectors are linearly independent, can it always be concluded they span \mathbb{R}^5 ?
6. If you have 6 vectors in \mathbb{R}^5 , is it possible they are linearly independent? Explain.
7. Let \mathbb{P}_3 be the polynomials of degree no more than 3. Determine which of the following are bases for this vector space.

- (a) $\{x + 1, x^3 + x^2 + 2x, x^2 + x, x^3 + x^2 + x\}$
 (b) $\{x^3 + 1, x^2 + x, 2x^3 + x^2, 2x^3 - x^2 - 3x + 1\}$

8. In the context of the above problem, consider polynomials

$$\{a_i x^3 + b_i x^2 + c_i x + d_i, i = 1, 2, 3, 4\}$$

Show that this collection of polynomials is linearly independent on an interval $[s, t]$ if and only if

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$$

is an invertible matrix.

9. Let the field of scalars be \mathbb{Q} , the rational numbers and let the vectors be of the form $a + b\sqrt{2}$ where a, b are rational numbers. Show that this collection of vectors is a vector space with field of scalars \mathbb{Q} and give a basis for this vector space.
10. Suppose V is a finite dimensional vector space. Based on the exchange theorem above, it was shown that any two bases have the same number of vectors in them. Give a different proof of this fact using the earlier material in the book. **Hint:** Suppose $\{\vec{x}_1, \dots, \vec{x}_n\}$ and $\{\vec{y}_1, \dots, \vec{y}_m\}$ are two bases with $m < n$. Then define

$$\phi : \mathbb{R}^n \mapsto V, \psi : \mathbb{R}^m \mapsto V$$

by

$$\phi(\vec{a}) = \sum_{k=1}^n a_k \vec{x}_k, \psi(\vec{b}) = \sum_{j=1}^m b_j \vec{y}_j$$

Consider the linear transformation, $\psi^{-1} \circ \phi$. Argue it is a one to one and onto mapping from \mathbb{R}^n to \mathbb{R}^m . Now consider a matrix of this linear transformation and its row reduced echelon form.

A. SOME PREREQUISITE TOPICS

The topics presented in this section are important concepts in mathematics and therefore should be examined.

A.1 SETS AND SET NOTATION

A set is a collection of things called elements. For example $\{1, 2, 3, 8\}$ would be a set consisting of the elements 1, 2, 3, and 8. To indicate that 3 is an element of $\{1, 2, 3, 8\}$, it is customary to write $3 \in \{1, 2, 3, 8\}$. We can also indicate when an element is not in a set, by writing $9 \notin \{1, 2, 3, 8\}$ which says that 9 is not an element of $\{1, 2, 3, 8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2. This would be written as $S = \{x \in \mathbb{Z} : x > 2\}$. This notation says: S is the set of all integers, x , such that $x > 2$.

Suppose A and B are sets with the property that every element of A is an element of B . Then we say that A is a subset of B . For example, $\{1, 2, 3, 8\}$ is a subset of $\{1, 2, 3, 4, 5, 8\}$. In symbols, we write $\{1, 2, 3, 8\} \subseteq \{1, 2, 3, 4, 5, 8\}$. It is sometimes said that “ A is contained in B ” or even “ B contains A ”. The same statement about the two sets may also be written as $\{1, 2, 3, 4, 5, 8\} \supseteq \{1, 2, 3, 8\}$.

We can also talk about the *union* of two sets, which we write as $A \cup B$. This is the set consisting of everything which is an element of at least one of the sets, A or B . As an example of the union of two sets, consider $\{1, 2, 3, 8\} \cup \{3, 4, 7, 8\} = \{1, 2, 3, 4, 7, 8\}$. This set is made up of the numbers which are in at least one of the two sets.

In general

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Notice that an element which is in *both* A and B is also in the union, as well as elements which are in only one of A or B .

Another important set is the intersection of two sets A and B , written $A \cap B$. This set consists of everything which is in *both* of the sets. Thus $\{1, 2, 3, 8\} \cap \{3, 4, 7, 8\} = \{3, 8\}$ because 3 and 8 are those elements the two sets have in common. In general,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If A and B are two sets, $A \setminus B$ denotes the set of things which are in A but not in B . Thus

$$A \setminus B = \{x \in A : x \notin B\}$$

For example, if $A = \{1, 2, 3, 8\}$ and $B = \{3, 4, 7, 8\}$, then $A \setminus B = \{1, 2, 3, 8\} \setminus \{3, 4, 7, 8\} = \{1, 2\}$.

A special set which is very important in mathematics is the empty set denoted by \emptyset . The empty set, \emptyset , is defined as the set which has no elements in it. It follows that the empty set is a subset of every set. This is true because if it were not so, there would have to exist a set A , such that \emptyset has something in it which is not in A . However, \emptyset has nothing in it and so it must be that $\emptyset \subseteq A$.

We can also use brackets to denote sets which are intervals of numbers. Let a and b be real numbers. Then

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$

These sorts of sets of real numbers are called intervals. The two points a and b are called endpoints, or bounds, of the interval. In particular, a is the *lower bound* while b is the *upper bound* of the above intervals, where applicable. Other intervals such as $(-\infty, b)$ are defined by analogy to what was just explained. In general, the curved parenthesis, $($, indicates the end point is not included in the interval, while the square parenthesis, $]$, indicates this end point is included. The reason that there will always be a curved parenthesis next to ∞ or $-\infty$ is that these are not real numbers and cannot be included in the interval in the way a real number can.

To illustrate the use of this notation relative to intervals consider three examples of inequalities. Their solutions will be written in the interval notation just described.

Example A.1: Solving an Inequality

Solve the inequality $2x + 4 \leq x - 8$.

Solution. We need to find x such that $2x + 4 \leq x - 8$. Solving for x , we see that $x \leq -12$ is the answer. This is written in terms of an interval as $(-\infty, -12]$. \square

Consider the following example.

Example A.2: Solving an Inequality

Solve the inequality $(x + 1)(2x - 3) \geq 0$.

Solution. We need to find x such that $(x + 1)(2x - 3) \geq 0$. The solution is given by $x \leq -1$ or $x \geq \frac{3}{2}$. Therefore, x which fit into either of these intervals gives a solution. In terms of set notation this is denoted by $(-\infty, -1] \cup [\frac{3}{2}, \infty)$. \square

Consider one last example.

Example A.3: Solving an Inequality

Solve the inequality $x(x + 2) \geq -4$.

Solution. This inequality is true for any value of x where x is a real number. We can write the solution as \mathbb{R} or $(-\infty, \infty)$. \square

In the next section, we examine another important mathematical concept.

A.2 WELL ORDERING AND INDUCTION

We begin this section with some important notation. Summation notation, written $\sum_{i=1}^j i$, represents a sum. Here, i is called the index of the sum, and we add iterations until $i = j$. For example,

$$\sum_{i=1}^j i = 1 + 2 + \cdots + j$$

Another example:

$$a_{11} + a_{12} + a_{13} = \sum_{i=1}^3 a_{1i}$$

The following notation is a specific use of summation notation.

Notation A.4: Summation Notation

Let a_{ij} be real numbers, and suppose $1 \leq i \leq r$ while $1 \leq j \leq s$. These numbers can be listed in a rectangular array as given by

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rs} \end{array}$$

Then $\sum_{j=1}^s \sum_{i=1}^r a_{ij}$ means to first sum the numbers in each column (using i as the index) and then to add the sums which result (using j as the index). Similarly, $\sum_{i=1}^r \sum_{j=1}^s a_{ij}$ means to sum the vectors in each row (using j as the index) and then to add the sums which result (using i as the index).

Notice that since addition is commutative, $\sum_{j=1}^s \sum_{i=1}^r a_{ij} = \sum_{i=1}^r \sum_{j=1}^s a_{ij}$.

We now consider the main concept of this section. Mathematical induction and well ordering are two extremely important principles in math. They are often used to prove significant things which would be hard to prove otherwise.

Definition A.5: Well Ordered

A set is well ordered if every nonempty subset S , contains a smallest element z having the property that $z \leq x$ for all $x \in S$.

In particular, the set of natural numbers defined as

$$\mathbb{N} = \{1, 2, \dots\}$$

is well ordered.

Consider the following proposition.

Proposition A.6: Well Ordered Sets

Any set of integers larger than a given number is well ordered.

This proposition claims that if a set has a lower bound which is a real number, then this set is well ordered.

Further, this proposition implies the principle of mathematical induction. The symbol \mathbb{Z} denotes the set of all integers. Note that if a is an integer, then there are no integers between a and $a + 1$.

Theorem A.7: Mathematical Induction

A set $S \subseteq \mathbb{Z}$, having the property that $a \in S$ and $n + 1 \in S$ whenever $n \in S$, contains all integers $x \in \mathbb{Z}$ such that $x \geq a$.

Proof. Let T consist of all integers larger than or equal to a which are not in S . The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T , denoted as b . It must be the case that $b > a$ since by definition, $a \notin T$. Thus $b \geq a + 1$, and so $b - 1 \geq a$ and $b - 1 \notin S$ because if $b - 1 \in S$, then $b - 1 + 1 = b \in S$ by the assumed property of S . Therefore, $b - 1 \in T$ which contradicts the choice of b as the smallest element of T . ($b - 1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that every integer at least as large as a is also in S . \square

Mathematical induction is a very useful device for proving theorems about the integers. The procedure is as follows.

Procedure A.8: Proof by Mathematical Induction

Suppose S_n is a statement which is a function of the number n , for $n = 1, 2, \dots$, and we wish to show that S_n is true for all $n \geq 1$. To do so using mathematical induction, use the following steps.

1. **Base Case:** Show S_1 is true.
2. Assume S_n is true for some n , which is the **induction hypothesis**. Then, using this assumption, show that S_{n+1} is true.

Proving these two steps shows that S_n is true for all $n = 1, 2, \dots$.

We can use this procedure to solve the following examples.

Example A.9: Proving by Induction

Prove by induction that $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution. By Procedure A.8, we first need to show that this statement is true for $n = 1$. When $n = 1$, the statement says that

$$\begin{aligned}\sum_{k=1}^1 k^2 &= \frac{1(1+1)(2(1)+1)}{6} \\ &= \frac{6}{6} \\ &= 1\end{aligned}$$

The sum on the left hand side also equals 1, so this equation is true for $n = 1$.

Now suppose this formula is valid for some $n \geq 1$ where n is an integer. Hence, the following equation is true.

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1.1}$$

We want to show that this is true for $n + 1$.

Suppose we add $(n + 1)^2$ to both sides of equation 1.1.

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^n k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\end{aligned}$$

The step going from the first to the second line is based on the assumption that the formula is true for n . Now simplify the expression in the second line,

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

This equals

$$(n+1) \left(\frac{n(2n+1)}{6} + (n+1) \right)$$

and

$$\frac{n(2n+1)}{6} + (n+1) = \frac{6(n+1) + 2n^2 + n}{6} = \frac{(n+2)(2n+3)}{6}$$

Therefore,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

showing the formula holds for $n+1$ whenever it holds for n . This proves the formula by mathematical induction. In other words, this formula is true for all $n = 1, 2, \dots$. \square

Consider another example.

Example A.10: Proving an Inequality by Induction

Show that for all $n \in \mathbb{N}$, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

Solution. Again we will use the procedure given in Procedure A.8 to prove that this statement is true for all n . Suppose $n = 1$. Then the statement says

$$\frac{1}{2} < \frac{1}{\sqrt{3}}$$

which is true.

Suppose then that the inequality holds for n . In other words,

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

is true.

Now multiply both sides of this inequality by $\frac{2n+1}{2n+2}$. This yields

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{2n+1}} \frac{2n+1}{2n+2} = \frac{\sqrt{2n+1}}{2n+2}$$

The theorem will be proved if this last expression is less than $\frac{1}{\sqrt{2n+3}}$. This happens if and only if

$$\left(\frac{1}{\sqrt{2n+3}} \right)^2 = \frac{1}{2n+3} > \frac{2n+1}{(2n+2)^2}$$

which occurs if and only if $(2n+2)^2 > (2n+3)(2n+1)$ and this is clearly true which may be seen from expanding both sides. This proves the inequality. \square

Let's review the process just used. If S is the set of integers at least as large as 1 for which the formula holds, the first step was to show $1 \in S$ and then that whenever $n \in S$, it follows $n + 1 \in S$. Therefore, by the principle of mathematical induction, S contains $[1, \infty) \cap \mathbb{Z}$, all positive integers. In doing an inductive proof of this sort, the set S is normally not mentioned. One just verifies the steps above.

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